Optimization and Simulation

Multi-objective optimization

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Multi-objective optimization

Concept

- Need for minimizing several objective functions.
- In many practical applications, the objectives are conflicting.
- Improving one objective may deteriorate several others.

Examples

- Transportation: maximize level of service, minimize costs.
- Finance: maximize return, minimize risk.
- Survey: maximize information, minimize number of questions (burden).

Multi-objective optimization

$$\min_x F(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_P(x) \end{pmatrix}$$

subject to

$$x \in \mathcal{F} \subseteq \mathbb{R}^n,$$

where

$$F : \mathbb{R}^n \to \mathbb{R}^P.$$

Outline

Dominance

Dominance

Consider $x_1, x_2 \in \mathbb{R}^n$. x_1 is dominating x_2 if

1. x_1 is no worse in any objective

$$\forall i \in \{1, \ldots, p\}, f_i(x_1) \leq f_i(x_2),$$

2. x_1 is strictly better in at least one objective

$$\exists i \in \{1, \ldots, p\}, f_i(x_1) < f_i(x_2).$$

Notation

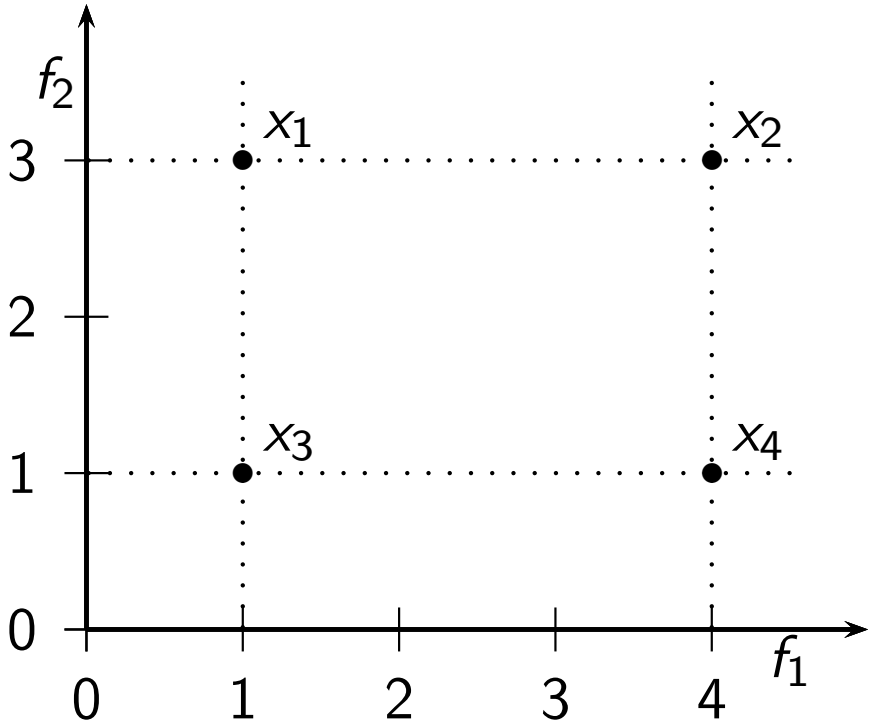
x_1 dominates x_2: $F(x_1) \prec F(x_2)$.

Dominance

Properties

- Not reflexive: $x \not\prec x$
- Not symmetric: $x \prec y \not\Rightarrow y \prec x$
- Instead: $x \prec y \Rightarrow y \not\prec x$
- Transitive: $x \prec y$ and $y \prec z \Rightarrow x \prec z$
- Not complete: $\exists x, y$: $x \not\prec y$ and $y \not\prec x$

Dominance: example



$$
\begin{aligned}
F(x_3) &\prec F(x_2) \\
F(x_3) &\prec F(x_1) \\
F(x_1) &\not\prec F(x_4) \\
F(x_4) &\not\prec F(x_1)
\end{aligned}
$$

Optimality

Pareto optimality

The vector $x^* \in \mathcal{F}$ is Pareto optimal if it is not dominated by any feasible solution:

$$\nexists x \in \mathcal{F} \text{ such that } F(x) \prec F(x^*).$$

Intuition

x^* is Pareto optimal if no objective can be improved without degrading at least one of the others.

Optimality

Weak Pareto optimality

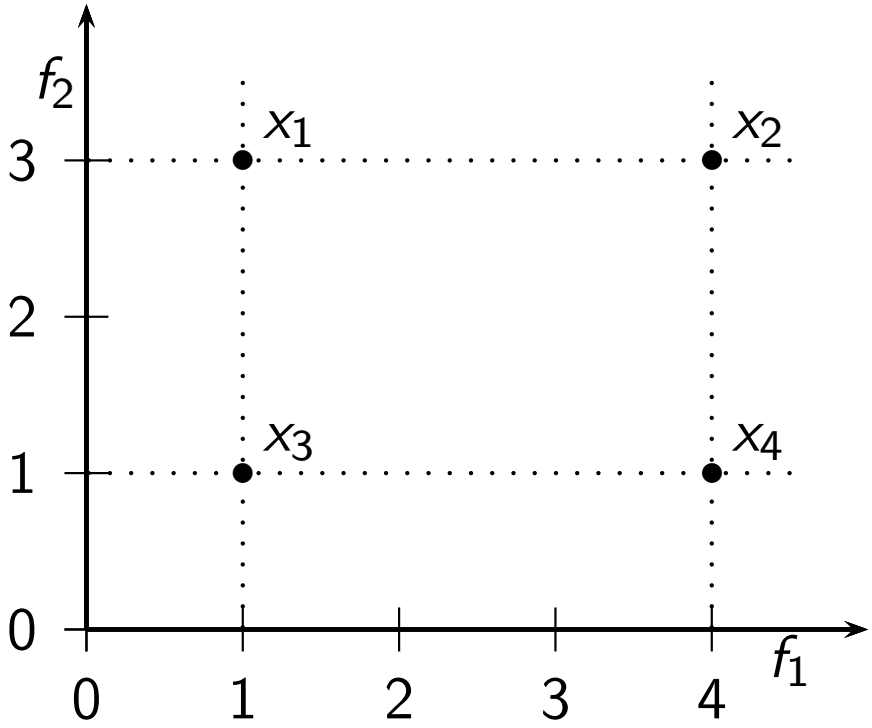
The vector $x^* \in \mathcal{F}$ is weakly Pareto optimal if there is no $x \in \mathcal{F}$ such that $\forall i = 1, \ldots, p$,

$$f_i(x) < f_i(x^*),$$

Pareto optimality

- P^*: set of Pareto optimal solutions
- WP^*: set of weakly Pareto optimal solutions
- $P^* \subseteq WP^* \subseteq \mathcal{F}$

Dominance: example



- x_3: Pareto optimal.
- x_1, x_3, x_4: weakly Pareto optimal.

Pareto frontier

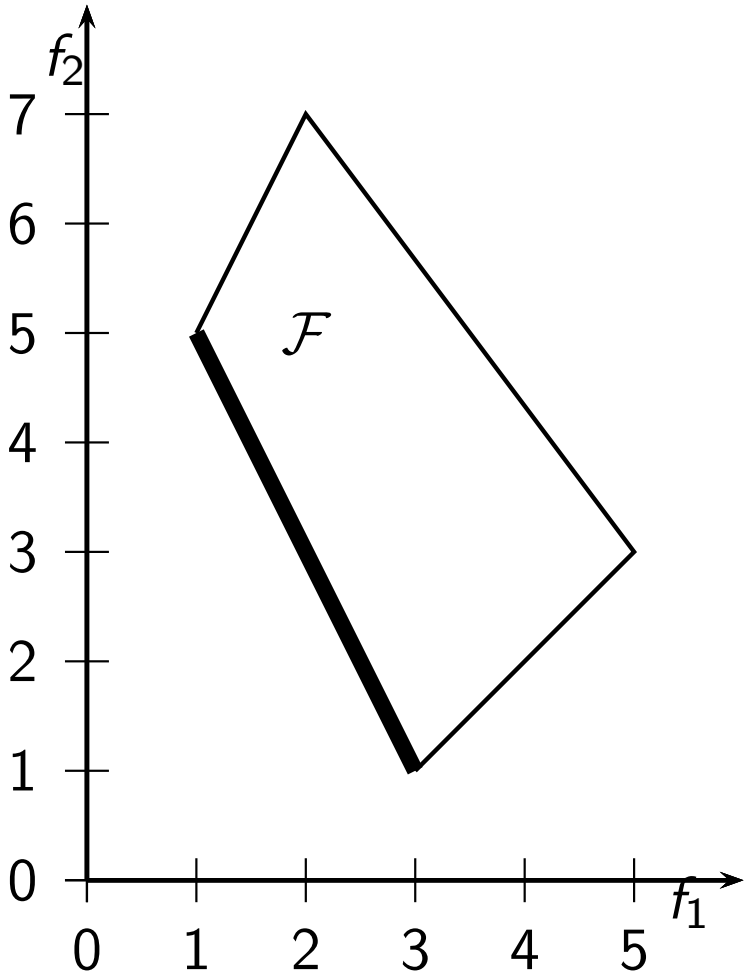
Pareto optimal set

$$P^* = \{x^* \in \mathcal{F} | \nexists x \in \mathcal{F} : F(x) \prec F(x^*)\}$$

Pareto frontier

$$PF^* = \{F(x^*) | x \in P^*\}$$

Pareto frontier



Outline

Weighted sum

Weights

For each $i = 1, \ldots, p$, $w_i > 0$ is the weight of objective i.

Optimization

$$\min_{x \in \mathcal{F}} \sum_{i=1}^{p} w_i f_i(x). \tag{1}$$

Comments

- Weights may be difficult to interpret in practice.
- Generates a Pareto optimal solution.
- In the convex case, if x^* is Pareto optimal, there exists a set of weights such that x^* is the solution of (1)

Weighted sum: example

Train service

- f_1: minimize travel time
- f_2: minimize number of trains
- f_3: maximize number of passengers

Definition of the weights

- Transform each objective into monetary costs.
- Travel time: use value-of-time.
- Number of trains: estimate the cost of running a train.
- Number of passengers: estimate the revenues generated by the passengers.

Goal programming

Goals

For each $i = 1, \ldots, p$, g_i is the "ideal" or "target" objective function defined by the modeler.

Optimization

$$\min_{x \in \mathcal{F}} \|F(x) - g\|_\ell = \sqrt[\ell]{\sum_{i=1}^{p} |F_i(x) - g_i|^\ell}$$

Issue

Not really optimizing the objectives

Outline

Lexicographic optimization

Sorted objective

Assume that the objectives are sorted from the most important ($i = 1$) to the least important ($i = p$).

First problem

$$f_1^* = \min_{x \in \mathcal{F}} f_1(x)$$

ℓth problem

$$f_\ell^* = \min f_\ell(x)$$

subject to

$$\begin{aligned} x &\in \mathcal{F} \\ f_i(x) &= f_i^*, \ i = 1, \ldots, \ell - 1. \end{aligned}$$

ε-lexicographic optimization

Sorted objective and tolerances

- Assume that the objectives are sorted from the most important ($i = 1$) to the least important ($i = p$).
- For each $i = 1, \ldots, p$, $\varepsilon_i \geq 0$ is a tolerance on the objective f_i.

First problem

$$f_1^* = \min_{x \in \mathcal{F}} f_1(x)$$

ℓth problem

$$f_\ell^* = \min f_\ell(x)$$

subject to

$$\begin{aligned} x &\in \mathcal{F} \\ f_i(x) &\leq f_i^* + \varepsilon_i, \ i = 1, \ldots, \ell - 1. \end{aligned}$$

Outline

ε-constraints formulation

Reference objective and upper bounds

- Select a reference objective $\ell \in \{1, \ldots, p\}$.
- Impose an upper bound ε_i on each other objective.

Constrained optimization

$$\min_{x \in \mathcal{F}} f_\ell(x)$$

subject to

$$f_i(x) \leq \varepsilon_i, \ i \neq \ell.$$

Property

If a solution exists, it is weakly Pareto optimal.

Outline

Local search

Main difference with single objective

Maintain a set $\mathcal{P}$ of potential Pareto optimal solutions

$$\forall x, y \in \mathcal{P}, F(x) \not\prec F(y) \text{ and } F(y) \not\prec F(x).$$

Initialization

Start with a first set $\mathcal{P}$ of candidate solutions.

Main iteration

- Select randomly x from $\mathcal{P}$ and consider x^+ a neighbor of x.
- Define
$$\mathcal{D}(x^+) = \{y \in \mathcal{P} \text{ such that } F(x^+) \prec F(y)\}.$$
- Define
$$\mathcal{S}(x^+) = \{y \in \mathcal{P} \text{ such that } F(y) \prec F(x^+)\}.$$

Local search

Main iteration

- If $\mathcal{S}(x^+) = \emptyset$

$$\mathcal{P}^+ = \mathcal{P} \cup \{x^+\} \setminus \mathcal{D}(x^+).$$

Property of $\mathcal{P}^+$

$$\forall x, y \in \mathcal{P}^+, F(x) \not\prec F(y) \text{ and } F(y) \not\prec F(x).$$

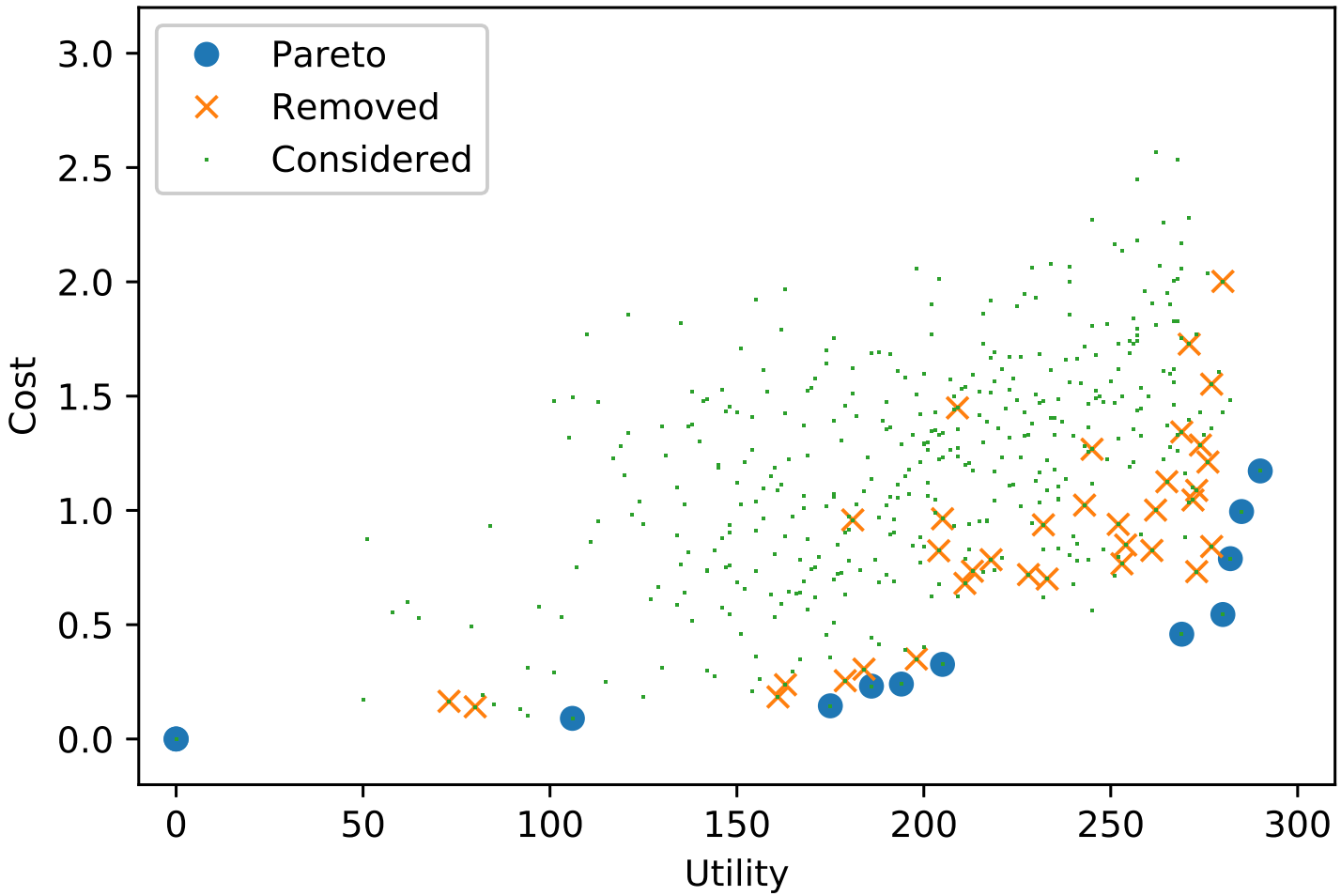
Proof

- For x, y different from x^+, already valid in $\mathcal{P}$.
- Consider x^+, $y \in \mathcal{P}^+$:
  - $y \in \mathcal{P}^+ \Rightarrow y \notin \mathcal{D}(x^+) \Rightarrow F(x^+) \not\prec F(y)$.
  - $x^+ \in \mathcal{P}^+ \Rightarrow \mathcal{S}(x^+) = \emptyset \Rightarrow y \notin \mathcal{S}(x^+) \Rightarrow F(y) \not\prec F(x^+)$.

Example: priced knapsack

Utility	Weight	Cost
80	84	0.50328447
31	27	0.41431774
48	47	0.07765353
17	22	0.75842330
27	21	0.14050556
84	96	0.72089439
34	42	0.11669739
39	46	0.56723896
46	54	0.02430532
58	53	0.01255171
23	32	0.03059062
67	78	0.17285314

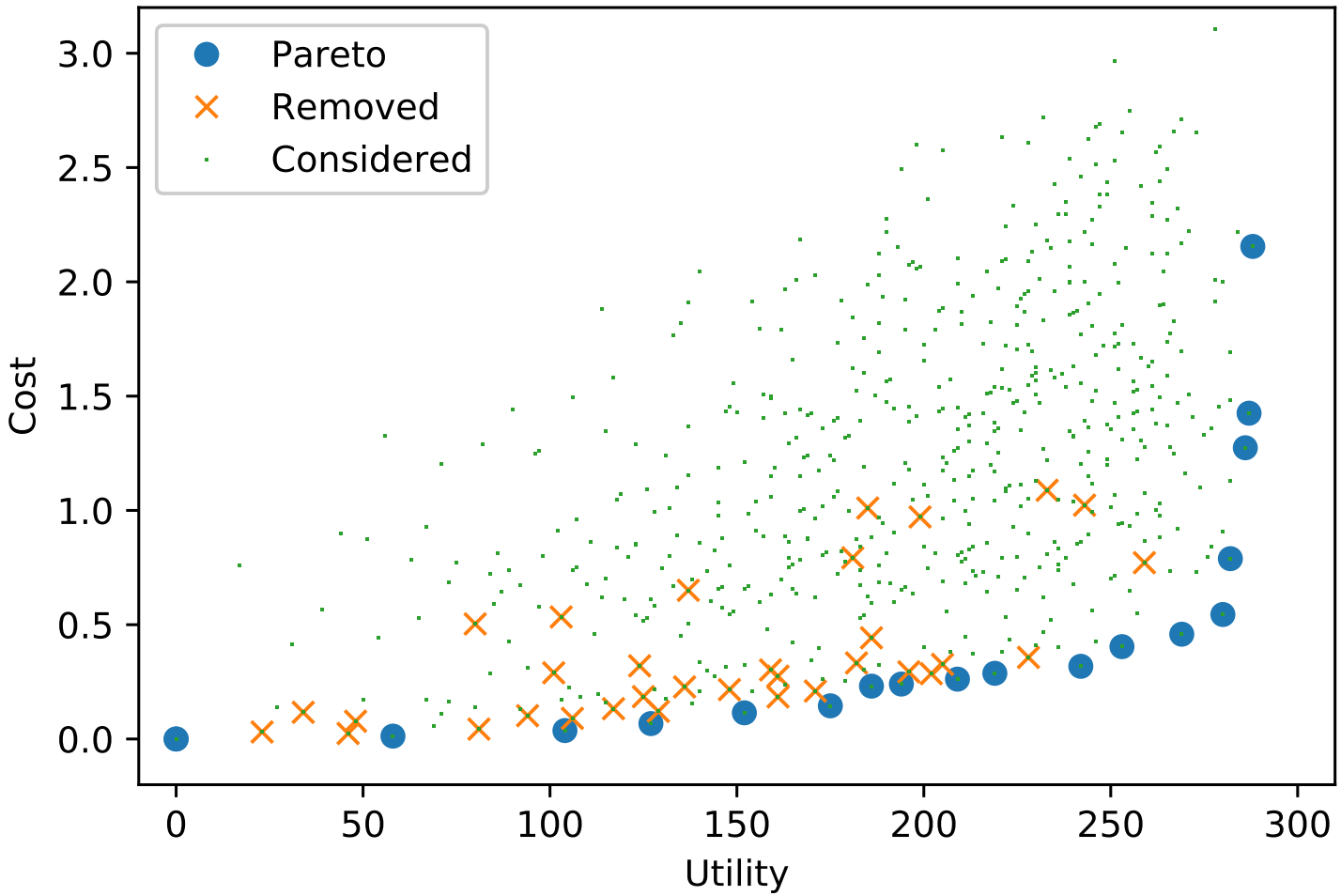
Example: local search with neighborhood $k = 4$



Variable Neighborhood Search

- Neighborhood of size 1 Pareto solutions: 1
- Neighborhood of size 2 Pareto solutions: 16
- Neighborhood of size 3 Pareto solutions: 16
- Neighborhood of size 4 Pareto solutions: 16
- Neighborhood of size 5 Pareto solutions: 16
- Neighborhood of size 6 Pareto solutions: 16
- Neighborhood of size 7 Pareto solutions: 18
- Neighborhood of size 8 Pareto solutions: 19
- Neighborhood of size 9 Pareto solutions: 19
- Neighborhood of size 10 Pareto solutions: 19
- Neighborhood of size 11 Pareto solutions: 19
- Neighborhood of size 12 Pareto solutions: 19
- Pareto solutions: 19

Variable Neighborhood Search



Conclusion

Problem definition

- Need for trade-offs.
- Concept of Pareto frontier.

Algorithms

- Heuristics.
- Most of time driven by problem knowledge.