

# Optimization and Simulation

## Simulating events: the Poisson process

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# Siméon Denis Poisson



Siméon-Denis Poisson

French mathematician (1781–1840).

# Outline

- 1 Poisson random variable
- 2 Poisson process
- 3 Non homogeneous Poisson process

# Binomial random variable

## Context

- $n$ : number of independent trials
- $p$ : probability of a success
- $X$ : number of successes

## Probability of $k$ successes

$$\Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$

# Poisson random variable

## Context: continuous trials

- $n \rightarrow +\infty$ : large number of trials
- $p \rightarrow 0$ : low probability of a success
- $np \rightarrow \lambda$ : success rate
- $X$ : number of successes

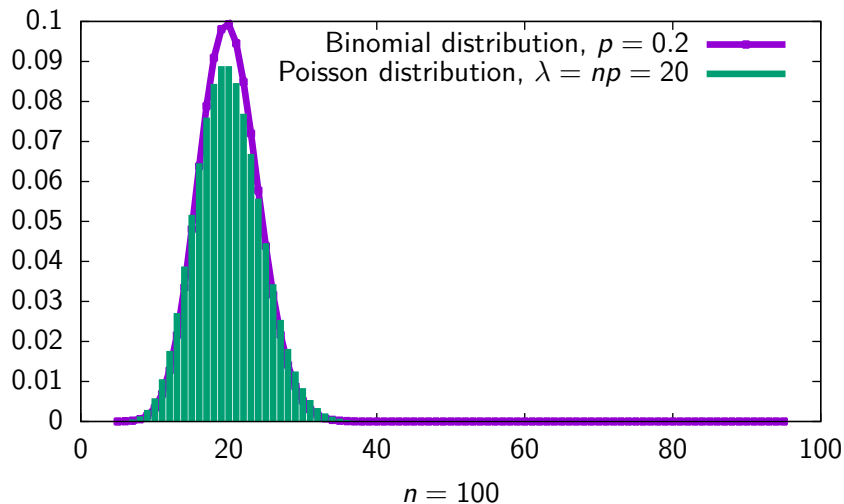
## Probability of $k$ successes

$$\Pr(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

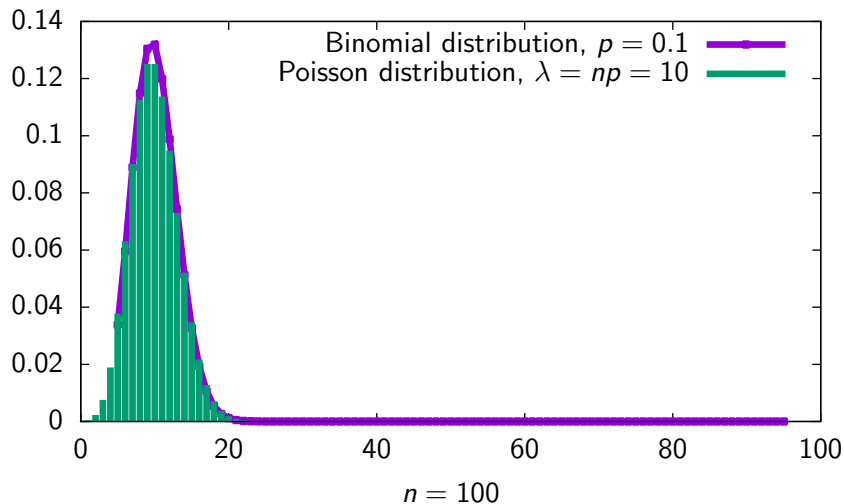
## Property

$$E[X] = \text{Var}(X) = \lambda.$$

## Poisson random variable



## Poisson random variable



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# Poisson process

Events are occurring at random time points

$N(t)$  is the number of events during  $[0, t]$

Poisson process with rate  $\lambda > 0$  if

- ①  $N(0) = 0$ ,
- ② # of events occurring in disjoint time intervals are independent,
- ③ distribution of  $N(t + s) - N(t)$  depends on  $s$ , not on  $t$ ,
- ④ probability of one event in a small interval is approx.  $\lambda h$ :

$$\lim_{h \rightarrow 0} \frac{\Pr(N(h) = 1)}{h} = \lambda,$$

- ⑤ probability of two events in a small interval is approx. 0:

$$\lim_{h \rightarrow 0} \frac{\Pr(N(h) \geq 2)}{h} = 0.$$

# Poisson process

## Property

$$N(t) \sim \text{Poisson}(\lambda t), \quad \Pr(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

## Inter-arrival times

- $S_k$  is the time when the  $k$ th event occurs,
- $X_k = S_k - S_{k-1}$  is the time elapsed between event  $k - 1$  and event  $k$ .
- $X_1 = S_1$
- Distribution of  $X_1$ :  $\Pr(X_1 > t) = \Pr(N(t) = 0) = e^{-\lambda t}$ .
- Distribution of  $X_k$ :

$$\begin{aligned} \Pr(X_k > t | S_{k-1} = s) &= \Pr(0 \text{ events in } ]s, s + t] | S_{k-1} = s) \\ &= \Pr(0 \text{ events in } ]s, s + t]) \\ &= e^{-\lambda t}. \end{aligned}$$

# Poisson process

## Inter-arrival times (ctd.)

Therefore, the CDF of  $X_k$  is, for any  $k$ ,

$$F(t) = \Pr(X_k \leq t) = 1 - \Pr(X_k > t) = 1 - e^{-\lambda t}.$$

The pdf is

$$f(t) = \frac{dF(t)}{dt} = \lambda e^{-\lambda t}.$$

# Poisson process

## Conclusion

The inter-arrival times  $X_1, X_2, \dots$  are independent and identically distributed exponential random variables with parameter  $\lambda$ , and mean  $1/\lambda$ .

## Simulation

- Simulation of event times of a Poisson process with rate  $\lambda$  until time  $T$ :
  - 1  $t = 0, k = 0$ .
  - 2 Draw  $r \sim U(0, 1)$ .
  - 3  $t = t + \ln(r)/\lambda$ .
  - 4 If  $t > T$ , STOP.
  - 5  $k = k + 1, S_k = t$ .
  - 6 Go to step 2.

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# Non homogeneous Poisson process

Rate varies with time

$\lambda(t)$ .

Non homogeneous Poisson process with rate  $\lambda(t)$  if

- ①  $N(0) = 0$
- ② # of events occurring in disjoint time intervals are independent,
- ③ probability of one event in a small interval is approx.  $\lambda(t)h$ :

$$\lim_{h \rightarrow 0} \frac{\Pr((N(t+h) - N(t)) = 1)}{h} = \lambda(t),$$

- ④ probability of two events in a small interval is approx. 0:

$$\lim_{h \rightarrow 0} \frac{\Pr((N(t+h) - N(t)) \geq 2)}{h} = 0.$$

# Non homogeneous Poisson process

## Mean value function

$$m(t) = \int_0^t \lambda(s) ds, \quad t \geq 0.$$

## Poisson distribution

$$N(t+s) - N(t) \sim \text{Poisson}(m(t+s) - m(t))$$

## Link with homogeneous Poisson process

- Consider a Poisson process with rate  $\lambda$ .
- If an event occurs at time  $t$ , count it with probability  $p(t)$ .
- The process of counted events is a non homogeneous Poisson process with rate  $\lambda(t) = \lambda p(t)$ .

# Non homogeneous Poisson process

## Proof

- ①  $N(0) = 0$  [OK]
- ② # of events occurring in disjoint time intervals are independent, [OK]
- ③ probability of one event in a small interval is approx.  $\lambda(t)h$ : [?]

$$\lim_{h \rightarrow 0} \frac{\Pr((N(t+h) - N(t)) = 1)}{h} = \lambda(t),$$

- ④ probability of two events in a small interval is approx. 0: [OK]

$$\lim_{h \rightarrow 0} \frac{\Pr((N(t+h) - N(t)) \geq 2)}{h} = 0.$$



# Non homogeneous Poisson process

## Proof (ctd.)

- $N(t)$  number of events of the non homogeneous process
- $N'(t)$  number of events of the underlying homogeneous process

$$\Pr((N(t+h) - N(t)) = 1)$$

$$\begin{aligned} &= \sum_{k=1}^{\infty} \Pr((N'(t+h) - N'(t)) = k, 1 \text{ is counted}) \\ &= \Pr((N'(t+h) - N'(t)) = 1, 1 \text{ is counted}) \\ &= \Pr((N'(t+h) - N'(t)) = 1) \Pr(1 \text{ is counted}) \\ &= \Pr(N'(h) = 1) \Pr(1 \text{ is counted}) \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\Pr((N(t+h) - N(t)) = 1)}{h} &= \lim_{h \rightarrow 0} \frac{\Pr(N'(h) = 1)}{h} \Pr(1 \text{ is counted}) \\ &= \lambda p(t). \end{aligned}$$

# Non homogeneous Poisson process

Simulation of event times of a non homogeneous Poisson process with rate  $\lambda(t)$  until time  $T$

- 1 Consider  $\lambda$  such that  $\lambda(t) \leq \lambda$ , for all  $t \leq T$ .
- 2  $t = 0, k = 0$ .
- 3 Draw  $r \sim U(0, 1)$ .
- 4  $t = t - \ln(r)/\lambda$ .
- 5 If  $t > T$ , STOP.
- 6 Generate  $s \sim U(0, 1)$ .
- 7 If  $s \leq \lambda(t)/\lambda$ , then  $k = k + 1, S_k = t$ .
- 8 Go to step 3.

# Summary

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- Poisson random variable
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## Comments

- Main assumption: events occur continuously and independently of one another
- Typical usage: arrivals of customers in a queue
- Easy to simulate