

Optimization and Simulation

Simulating events: the Poisson process

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Siméon Denis Poisson



Siméon-Denis Poisson

French mathematician (1781–1840).

Outline

- 1 Poisson random variable
- 2 Poisson process
- 3 Non homogeneous Poisson process

Binomial random variable

Context

- n : number of independent trials
- p : probability of a success
- X : number of successes

Probability of k successes

$$\Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$

Poisson random variable

Context: continuous trials

- $n \rightarrow +\infty$: large number of trials
- $p \rightarrow 0$: low probability of a success
- $np \rightarrow \lambda$: success rate
- X : number of successes

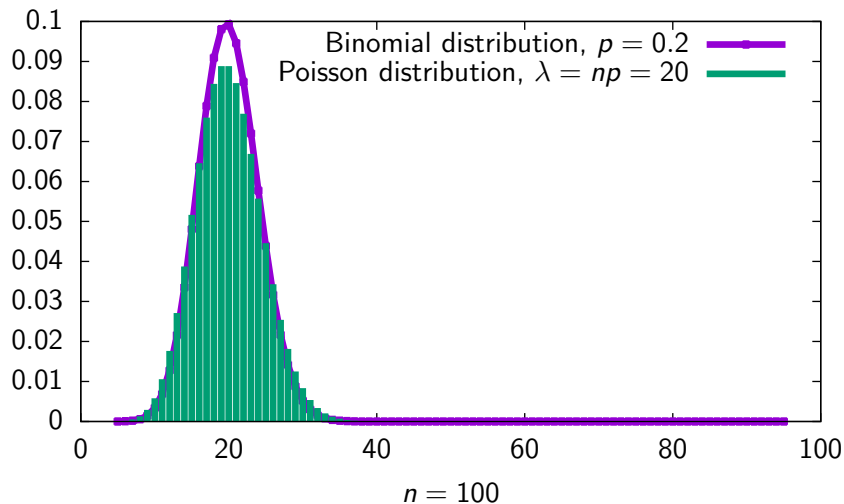
Probability of k successes

$$\Pr(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

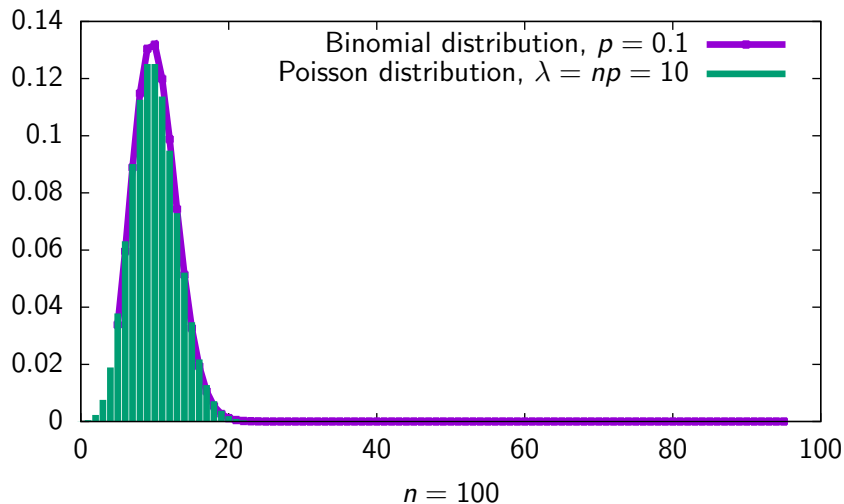
Property

$$E[X] = \text{Var}(X) = \lambda.$$

Poisson random variable



Poisson random variable



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Poisson process

Events are occurring at random time points

$N(t)$ is the number of events during $[0, t]$

Poisson process with rate $\lambda > 0$ if

- ① $N(0) = 0$,
- ② # of events occurring in disjoint time intervals are independent,
- ③ distribution of $N(t + s) - N(t)$ depends on s , not on t ,
- ④ probability of one event in a small interval is approx. λh :

$$\lim_{h \rightarrow 0} \frac{\Pr(N(h) = 1)}{h} = \lambda,$$

- ⑤ probability of two events in a small interval is approx. 0:

$$\lim_{h \rightarrow 0} \frac{\Pr(N(h) \geq 2)}{h} = 0.$$

Poisson process

Property

$$N(t) \sim \text{Poisson}(\lambda t), \quad \Pr(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

Inter-arrival times

- S_k is the time when the k th event occurs,
- $X_k = S_k - S_{k-1}$ is the time elapsed between event $k - 1$ and event k .
- $X_1 = S_1$
- Distribution of X_1 : $\Pr(X_1 > t) = \Pr(N(t) = 0) = e^{-\lambda t}$.
- Distribution of X_k :

$$\begin{aligned} \Pr(X_k > t | S_{k-1} = s) &= \Pr(0 \text{ events in }]s, s + t] | S_{k-1} = s) \\ &= \Pr(0 \text{ events in }]s, s + t]) \\ &= e^{-\lambda t}. \end{aligned}$$

Poisson process

Inter-arrival times (ctd.)

Therefore, the CDF of X_k is, for any k ,

$$F(t) = \Pr(X_k \leq t) = 1 - \Pr(X_k > t) = 1 - e^{-\lambda t}.$$

The pdf is

$$f(t) = \frac{dF(t)}{dt} = \lambda e^{-\lambda t}.$$

Poisson process

Conclusion

The inter-arrival times X_1, X_2, \dots are independent and identically distributed exponential random variables with parameter λ , and mean $1/\lambda$.

Simulation

- Simulation of event times of a Poisson process with rate λ until time T :
 - 1 $t = 0, k = 0$.
 - 2 Draw $r \sim U(0, 1)$.
 - 3 $t = t + \ln(r)/\lambda$.
 - 4 If $t > T$, STOP.
 - 5 $k = k + 1, S_k = t$.
 - 6 Go to step 2.

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Non homogeneous Poisson process

Rate varies with time

$\lambda(t)$.

Non homogeneous Poisson process with rate $\lambda(t)$ if

- ① $N(0) = 0$
- ② # of events occurring in disjoint time intervals are independent,
- ③ probability of one event in a small interval is approx. $\lambda(t)h$:

$$\lim_{h \rightarrow 0} \frac{\Pr((N(t+h) - N(t)) = 1)}{h} = \lambda(t),$$

- ④ probability of two events in a small interval is approx. 0:

$$\lim_{h \rightarrow 0} \frac{\Pr((N(t+h) - N(t)) \geq 2)}{h} = 0.$$

Non homogeneous Poisson process

Mean value function

$$m(t) = \int_0^t \lambda(s) ds, \quad t \geq 0.$$

Poisson distribution

$$N(t+s) - N(t) \sim \text{Poisson}(m(t+s) - m(t))$$

Link with homogeneous Poisson process

- Consider a Poisson process with rate λ .
- If an event occurs at time t , count it with probability $p(t)$.
- The process of counted events is a non homogeneous Poisson process with rate $\lambda(t) = \lambda p(t)$.

Non homogeneous Poisson process

Proof

- ① $N(0) = 0$ [OK]
- ② # of events occurring in disjoint time intervals are independent, [OK]
- ③ probability of one event in a small interval is approx. $\lambda(t)h$: [?]

$$\lim_{h \rightarrow 0} \frac{\Pr((N(t+h) - N(t)) = 1)}{h} = \lambda(t),$$

- ④ probability of two events in a small interval is approx. 0: [OK]

$$\lim_{h \rightarrow 0} \frac{\Pr((N(t+h) - N(t)) \geq 2)}{h} = 0.$$

Non homogeneous Poisson process

Proof (ctd.)

- $N(t)$ number of events of the non homogeneous process
- $N'(t)$ number of events of the underlying homogeneous process

$$\Pr((N(t+h) - N(t)) = 1)$$

$$\begin{aligned} &= \sum_{k=1}^{\infty} \Pr((N'(t+h) - N'(t)) = k, 1 \text{ is counted}) \\ &= \Pr((N'(t+h) - N'(t)) = 1, 1 \text{ is counted}) \\ &= \Pr((N'(t+h) - N'(t)) = 1) \Pr(1 \text{ is counted}) \\ &= \Pr(N'(h) = 1) \Pr(1 \text{ is counted}) \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\Pr((N(t+h) - N(t)) = 1)}{h} &= \lim_{h \rightarrow 0} \frac{\Pr(N'(h) = 1)}{h} \Pr(1 \text{ is counted}) \\ &= \lambda p(t). \end{aligned}$$

Non homogeneous Poisson process

Simulation of event times of a non homogeneous Poisson process with rate $\lambda(t)$ until time T

- 1 Consider λ such that $\lambda(t) \leq \lambda$, for all $t \leq T$.
- 2 $t = 0, k = 0$.
- 3 Draw $r \sim U(0, 1)$.
- 4 $t = t - \ln(r)/\lambda$.
- 5 If $t > T$, STOP.
- 6 Generate $s \sim U(0, 1)$.
- 7 If $s \leq \lambda(t)/\lambda$, then $k = k + 1, S_k = t$.
- 8 Go to step 3.

Summary

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- Poisson random variable
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Comments

- Main assumption: events occur continuously and independently of one another
- Typical usage: arrivals of customers in a queue
- Easy to simulate