Optimization and Simulation

Multi-objective optimization

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Multi-objective optimization

Concept

- Need for minimizing several objective functions.
- In many practical applications, the objectives are conflicting.
- Improving one objective may deteriorate several others.

Examples

- Transportation: maximize level of service, minimize costs.
- Finance: maximize return, minimize risk.
- Survey: maximize information, minimize number of questions (burden).

Multi-objective optimization

$$\min_x F(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_P(x) \end{pmatrix}$$

subject to

$$x \in \mathcal{F} \subseteq \mathbb{R}^n,$$

where

$$F : \mathbb{R}^n \to \mathbb{R}^p.$$

Outline

Dominance

Dominance

Consider $x_1, x_2 \in \mathbb{R}^n$. x_1 is dominating x_2 if

1. x_1 is no worse in any objective

$$\forall i \in \{1, \ldots, p\}, f_i(x_1) \leq f_i(x_2),$$

2. x_1 is strictly better in at least one objective

$$\exists i \in \{1, \ldots, p\}, f_i(x_1) < f_i(x_2).$$

Notation

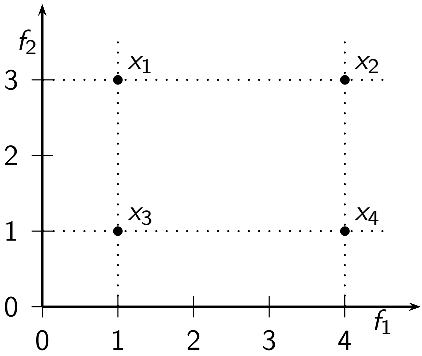
x_1 dominates x_2: $F(x_1) \prec F(x_2)$.

Dominance

Properties

- Not reflexive: $x \not\prec x$
- Not symmetric: $x \prec y \not\Rightarrow y \prec x$
- Instead: $x \prec y \Rightarrow y \not\prec x$
- Transitive: $x \prec y$ and $y \prec z \Rightarrow x \prec z$
- Not complete: $\exists x, y$: $x \not\prec y$ and $y \not\prec x$

Dominance: example



$$
\begin{array}{ccc}
F(x_3) & \prec & F(x_2) \\
F(x_3) & \prec & F(x_1) \\
F(x_1) & \not\prec & F(x_4) \\
F(x_4) & \not\prec & F(x_1)
\end{array}
$$

Optimality

Pareto optimality

The vector $x^* \in \mathcal{F}$ is Pareto optimal if it is not dominated by any feasible solution:

$$\nexists x \in \mathcal{F} \text{ such that } F(x) \prec F(x^*).$$

Intuition

x is Pareto optimal if no objective can be improved without degrading at least one of the others.

Optimality

Weak Pareto optimality

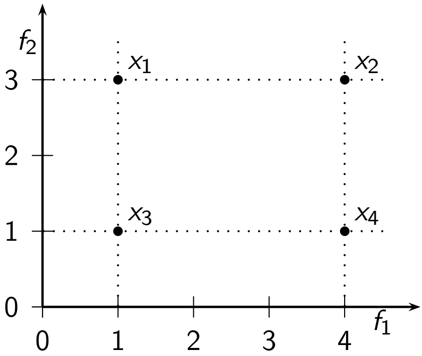
The vector $x^* \in \mathcal{F}$ is weakly Pareto optimal if there is no $x \in \mathcal{F}$ such that $\forall i = 1, \ldots, p$,

$$f_i(x) < f_i(x^*),$$

Pareto optimality

- P^*: set of Pareto optimal solutions
- WP^*: set of weakly Pareto optimal solutions
- $P^* \subseteq WP^* \subseteq \mathcal{F}$

Dominance: example



- x_3: Pareto optimal.
- x_1, x_3, x_4: weakly Pareto optimal.

Pareto frontier

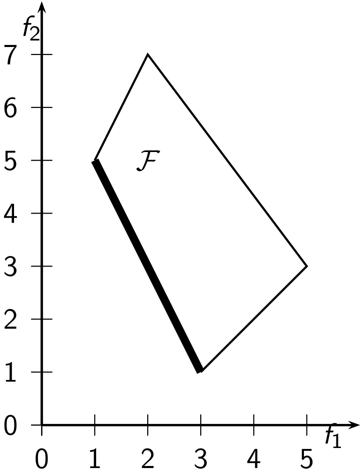
Pareto optimal set

$$P^* = \{x^* \in \mathcal{F} | \nexists x \in \mathcal{F} : F(x) \prec F(x^*)\}$$

Pareto frontier

$$PF^* = \{F(x^*) | x \in P^*\}$$

Pareto frontier



Outline

1 Definitions

2 Transformations into single-objective

3 Lexicographic rules

4 Constrained optimization

Weighted sum

Weights

For each $i = 1, \ldots, p$, $w_i > 0$ is the weight of objective i.

Optimization

$$\min_{x \in \mathcal{F}} \sum_{i=1}^{p} w_i f_i(x). \qquad (1)$$

Comments

- Weights may be difficult to interpret in practice.
- Generates a Pareto optimal solution.
- In the convex case, if x^* is Pareto optimal, there exists a set of weights such that x^* is the solution of (1)

Weighted sum: example

Train service

- f_1: minimize travel time
- f_2: minimize number of trains
- f_3: maximize number of passengers

Definition of the weights

- Transform each objective into monetary costs.
- Travel time: use value-of-time.
- Number of trains: estimate the cost of running a train.
- Number of passengers: estimate the revenues generated by the passengers.

Goal programming

Goals

For each $i = 1, \ldots, p$, g_i is the "ideal" or "target" objective function defined by the modeler.

Optimization

$$\min_{x \in \mathcal{F}} \|F(x) - g\|_\ell = \sqrt[\ell]{\sum_{i=1}^{p} |F_i(x) - g_i|^\ell}$$

Issue

Not really optimizing the objectives

Outline

Lexicographic optimization

Sorted objective

Assume that the objectives are sorted from the most important ($i = 1$) to the least important ($i = p$).

First problem

$$f_1^* = \min_{x \in \mathcal{F}} f_1(x)$$

ℓth problem

$$f_\ell^* = \min f_\ell(x)$$

subject to

$$\begin{aligned} x &\in \mathcal{F} \\ f_i(x) &= f_i^*, \; i = 1, \ldots, \ell - 1. \end{aligned}$$

ε-lexicographic optimization

Sorted objective and tolerances

- Assume that the objectives are sorted from the most important ($i = 1$) to the least important ($i = p$).
- For each $i = 1, \ldots, p$, $\varepsilon_i \geq 0$ is a tolerance on the objective f_i.

First problem

$$f_1^* = \min_{x \in \mathcal{F}} f_1(x)$$

ℓth problem

$$f_\ell^* = \min f_\ell(x)$$

subject to

$$\begin{aligned} x &\in \mathcal{F} \\ f_i(x) &\leq f_i^* + \varepsilon_i, \; i = 1, \ldots, \ell - 1. \end{aligned}$$

Outline

ε-constraints formulation

Reference objective and upper bounds

- Select a reference objective $\ell \in \{1, \ldots, p\}$.
- Impose an upper bound ε_i on each other objective.

Constrained optimization

$$\min_{x \in \mathcal{F}} f_\ell(x)$$

subject to

$$f_i(x) \leq \varepsilon_i, \ i \neq \ell.$$

Property

If a solution exists, it is weakly Pareto optimal.

Conclusion

Problem definition

- Need for trade-offs.
- Concept of Pareto frontier.

Algorithms

- Heuristics.
- Most of time driven by problem knowledge.