Sequential Quadratic Programming Method

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General description of the algorithm

Objective

Find the local minimum of the non-linear optimization problem:

$$\min_{x \in \mathcal{R}^n} f(x)$$

such that $h(x) = 0$.

Input

- Function $f : \mathcal{R}^n \to \mathcal{R}$, two-times differentiable
- Gradient $\nabla f : \mathcal{R}^n \to \mathcal{R}^n$
- Hessian $\nabla^2 f : \mathcal{R}^n \to \mathcal{R}^{n \times n}$
- Function $h : \mathcal{R}^n \to \mathcal{R}^m$, two-times differentiable
- Gradient $\nabla h : \mathcal{R}^n \to \mathcal{R}^{n \times m}$
- Hessian $\nabla^2 h_i : \mathcal{R}^n \to \mathcal{R}^{n \times n}$ for every constraint $i = 1, \ldots, m$
- Parameter $0 < \beta_1 < 1$
- Parameter $\bar{c} > 0$
- Initial solution (x_0, λ_0)
- Precision $\epsilon \in \mathcal{R}$, $\epsilon > 0$

Initialization

$k = 0$, $c_0 = \|\lambda_0\|_\infty + \bar{c}$

Iterations

1. Compute $\nabla^2_{xx}L(x_k,\lambda_k) = \nabla^2 f(x_k) + \sum_{i=1}^m (\lambda_k)_i \nabla^2 h_i(x_k)$
2. Find a positive-definite approximation H_k of $\nabla^2_{xx}L(x_k,\lambda_k)$ by using the modified Cholesky factorization
3. Find d_x, d_λ by solving the following quadratic problem

$$\min_d \nabla f(x_k)^T d + \frac{1}{2} d^T \nabla^2_{xx} L(x_k,\lambda_k) d$$

$$s.t. \quad \nabla h(x_k)^T d + h(x_k) = 0$$

   by using the analytic solution,

$$d_\lambda = H^{-1}(h(x_k) - \nabla h(x_k)^T \nabla^2_{xx} L(x_k,\lambda_k)^{-1} \nabla f(x_k))$$

   with $H = \nabla h(x_k)^T \nabla^2_{xx} L(x_k,\lambda_k)^{-1} \nabla h(x_k)$

$$d_x = -\nabla^2_{xx} L(x_k,\lambda_k)^{-1} (\nabla h(x_k) d_\lambda + \nabla f(x_k))$$

4. $c^+ = \|d_\lambda\|_\infty + \bar{c}$
5. Update the penalty parameter:
   (a) if $c_k \geq 1.1c^+$, then $c_{k+1} = \frac{1}{2}(c_k + c^+)$;
   (b) if $c^+ \leq c_k < 1.1c^+$, then $c_{k+1} = c_k$;
   (c) if $c_k < c^+$, then $c_{k+1} = \max\{1.5c_k, c^+\}$.
6. Compute $\phi'_{c_k}(x_k, d_x) = \nabla f(x_k)^T d_x - c_k \|h(x_k)\|_1$
   (a) $i = 0$, $\alpha_0 = 1$;
   (b) until $\phi_{c_k}(x_k + \alpha_i d_x) < \phi_{c_k}(x_k) + \alpha_i \beta_1 \phi'_{c_k}(x_k, d_x)$, set $\alpha_{i+1} = \alpha_i/2$ and $i = i + 1$;
   (c) $\alpha = \alpha_i$.
7. $x_{k+1} = x_k + \alpha d_x$
8. $\lambda_{k+1} = d_\lambda$
9. $k = k + 1$

Stopping criterion

$\|\nabla L(x_k,\lambda_k)\| \leq \epsilon$

Algorithm testing & analysis

The sutdents will implement and apply the above algorithm to the following problem:

$$\min_{x \in \mathcal{R}^{10}} \quad \sum_{i=1}^{10} e^{x_i}(c_i + x_i - \ln(\sum_{k=1}^{10} e^{x_k}))$$

$$e^{x_1} + 2e^{x_2} + 2e^{x_3} + e^{x_6} + e^{x_{10}} = 2$$

$$e^{x_4} + 2e^{x_5} + e^{x_6} + e^{x_7} = 1$$

$$e^{x_3} + e^{x_7} + e^{x_8} + 2e^{x_9} + e^{x_{10}} = 1$$

$$-100 \leq x_i \leq 100, \forall i = 1, \ldots, 10$$

with

$c_1 = -6.089$	$c_2 = -17.164$	$c_3 = -34.054$	$c_4 = -5.914$	$c_5 = -24.721$
$c_6 = -14.986$	$c_7 = -24.1$	$c_8 = -10.708$	$c_9 = -26.662$	$c_{10} = -22.179$

Suggested starting point: $x_0 = [-2.3, -2.3, \ldots, -2.3]$.

Note that the students need to transform the above problem to the required format, i.e., $\min_{x \in \mathcal{R}^n} f(x)$ s.t., $h(x) = 0$. It is encouraged that the students to change the value of β_1 (e.g., $\beta_1 = 0.2, 0.4, 0.6, 0.8$) and analyze its impact on the algorithm.