Interior Point Method

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General description of the algorithm

Objective

Find the global minimum of a linear optimization problem in standard form:

$$\min_{x \in X \subseteq \mathcal{R}^n} c^T x$$

where $X = \{x \mid Ax = b, x \geq 0\}$.

Input

- Matrix $A \in \mathcal{R}^{m \times n}$
- Vector $b \in \mathcal{R}^m$
- Cost vector $c \in \mathcal{R}^n$
- Initial solution $(x_0, \lambda_0, \mu_0)^T$, s.t., $Ax_0 = b$, $A^T \lambda_0 + \mu_0 = c$, $x_0 > 0$, $\mu_0 > 0$
- Initial value for the barrier's height $\epsilon_0 > 0$
- Precision $\bar{\epsilon} \in \mathcal{R}$, $\bar{\epsilon} > 0$

Iterations

1. Compute (d_x, d_λ, d_μ) by solving:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S_k & 0 & X_k \end{bmatrix} \begin{bmatrix} d_x \\ d_\lambda \\ d_\mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -X_k S_k e + \epsilon_k e \end{bmatrix}$$

where $e = (1, 1, \ldots, 1)^T$ and

$$X_k = \begin{bmatrix} x_{k,1} & 0 & \ldots & 0 & 0 \\ 0 & x_{k,2} & \ldots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \ldots & x_{k,n-1} & 0 \\ 0 & 0 & 0 & 0 & x_{k,n} \end{bmatrix}, S_k = \begin{bmatrix} \mu_{k,1} & 0 & \ldots & 0 & 0 \\ 0 & \mu_{k,2} & \ldots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \ldots & \mu_{k,n-1} & 0 \\ 0 & 0 & 0 & 0 & \mu_{k,n} \end{bmatrix}$$

2. Compute a step $0 < \alpha_k \leq 1$ such that

$$(x_{k+1}, \lambda_{k+1}, \mu_{k+1})^T = (x_k, \lambda_k, \mu_k)^T + \alpha_k (d_x, d_\lambda, d_\mu)^T$$

is strictly feasible, i.e., $(x_{k+1}, \lambda_{k+1}, \mu_{k+1}) \in \mathcal{S}$, where

$$\mathcal{S} = \{(x, \lambda, \mu) \mid Ax = b, A^T \lambda + \mu = c, x \geq 0, \mu \geq 0\}$$

3. Update the barrier's height by defining ϵ_{k+1}

4. $k = k + 1$

Stopping criterion

$\frac{1}{n} x_k^T \mu_k \leq \bar{\epsilon}$

Algorithm testing & analysis

The sutdents will implement and apply the above algorithm to the following problem (with $n = 10$):

$$\min \quad -\sum_{i=1}^{n} 2^{n-i} x_i$$

$$x_1 \leq 5$$

$$4x_1 + x_2 \leq 25$$

$$8x_1 + 4x_2 + x_3 \leq 125$$

$$\vdots$$

$$2^n x_1 + 2^{n-1} x_2 + \ldots + 4x_{n-1} + x_n \leq 5^n$$

$$x_1, x_2, \ldots, x_n \geq 0$$

Please describe your procedures to obtain an initial solution $(x_0, \lambda_0, \mu_0)^T$ and it is encouraged that the students to test the algorithm by adopting different barrier height updating strategies (satisfying $0 < \epsilon_{k+1} < \epsilon_k$ and $\lim_k \epsilon_k = 0$). Besides, please give an intuitive explaination on why your obtained solutions is optimal. In other words, can you solve the above problem without resorting to any solving algorithm?