Optimization and Simulation

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Introduction

- Management of complex systems
  - Transportation systems
  - Environmental systems
  - Process systems
  - Structural systems
  - `en.wikipedia.org/wiki/List_of_types_of_systems_engineering`
- The whole may be different from the sum of the parts
- Need for methods to deal with the complexity

- *To optimize*: to find the best configuration
- *To simulate*: to act like.

Optimization: the problem

$$\min_{x \in \mathbb{R}^n} f(x)$$

subject to

$$\begin{array}{rcl} h(x) & = & 0 \\ g(x) & \leq & 0 \\ x & \in & X \subseteq \mathbb{R}^n \end{array}$$

Modeling elements:

1. Decision variables: x
2. Objective function: $f : \mathbb{R}^n \to \mathbb{R}$ ($n > 0$)
3. Constraints:
   - equality: $h : \mathbb{R} \to \mathbb{R}^m$ ($m \geq 0$)
   - inequality: $g : \mathbb{R}^n \to \mathbb{R}^p$ ($p \geq 0$)
   - X is a convex set

The problem

- x_i, $i = 1, \ldots, n$, are continuous variables
- f, g and h are sufficiently differentiable
- $Y = \{x \in \mathbb{R}^n | h(x) = 0, g(x) \leq 0$ and $x \in X\}$ is non empty

Local minimum $x^* \in Y$ is a local minimum of the above problem if there exists $\varepsilon > 0$ such that

$$f(x^*) \leq f(x) \quad \forall x \in Y \text{ such that } \|x - x^*\| < \varepsilon.$$

Global minimum $x^* \in Y$ is a global minimum of the above problem if

$$f(x^*) \leq f(x) \quad \forall x \in Y.$$

Lagrangian

- Assume $X = \mathbb{R}^n$ in the above problem
- Consider $\lambda \in \mathbb{R}^m$
- Consider $\mu \in \mathbb{R}^p$

The function $L : \mathbb{R}^{n+m+p} \to \mathbb{R}$ defined as

$$\begin{aligned} L(x,\lambda,\mu) &= f(x) + \lambda^T h(x) + \mu^T g(x) \\ &= f(x) + \textstyle\sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^p \mu_j g_j(x) \end{aligned}$$

is called the lagrangian function.

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Dual function

- The function $q : \mathbb{R}^{m+p} \rightarrow \mathbb{R}$ defined as

$$q(\lambda, \mu) = \min_{x \in \mathbb{R}^n} L(x, \lambda, \mu)$$

  is called the dual function of the optimization problem.
- Parameters λ and μ are called dual variables. x are called primal variables.
- If x^* is a global minimum of the optimization problem, then, for any $\lambda \in \mathbb{R}^m$ and any $\mu \in \mathbb{R}$, $\mu \geq 0$, we have

$$q(\lambda, \mu) \leq f(x^*).$$

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Dual problem

Let $X_q \subseteq \mathbb{R}^{m+p}$ be the domain of q, that is

$$X_q = \{\lambda, \mu | q(\lambda, \mu) > -\infty\}$$

The optimization problem

$$\max_{\lambda,\mu} q(\lambda, \mu)$$

subject to

$$\mu \geq 0$$

and

$$(\lambda, \mu) \in X_q$$

is called the dual problem of the original problem, which is called the primal problem in this context.

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Duality results

Weak duality theorem Let x^* be a global minimum of the primal problem, and (λ^*, μ^*) a global maximum of the dual problem. Then,

$$q(\lambda^*, \mu^*) \leq f(x^*).$$

Convexity-concavity of the dual problem

- The objective function of the dual problem is concave.
- The feasible set of the dual problem is convex.