Simulating events: the Poisson process

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Siméon Denis Poisson

Siméon-Denis Poisson (1781–1840). French mathematician.

- Poisson random variable
- Poisson process
- Non homogeneous Poisson process

Poisson random variable

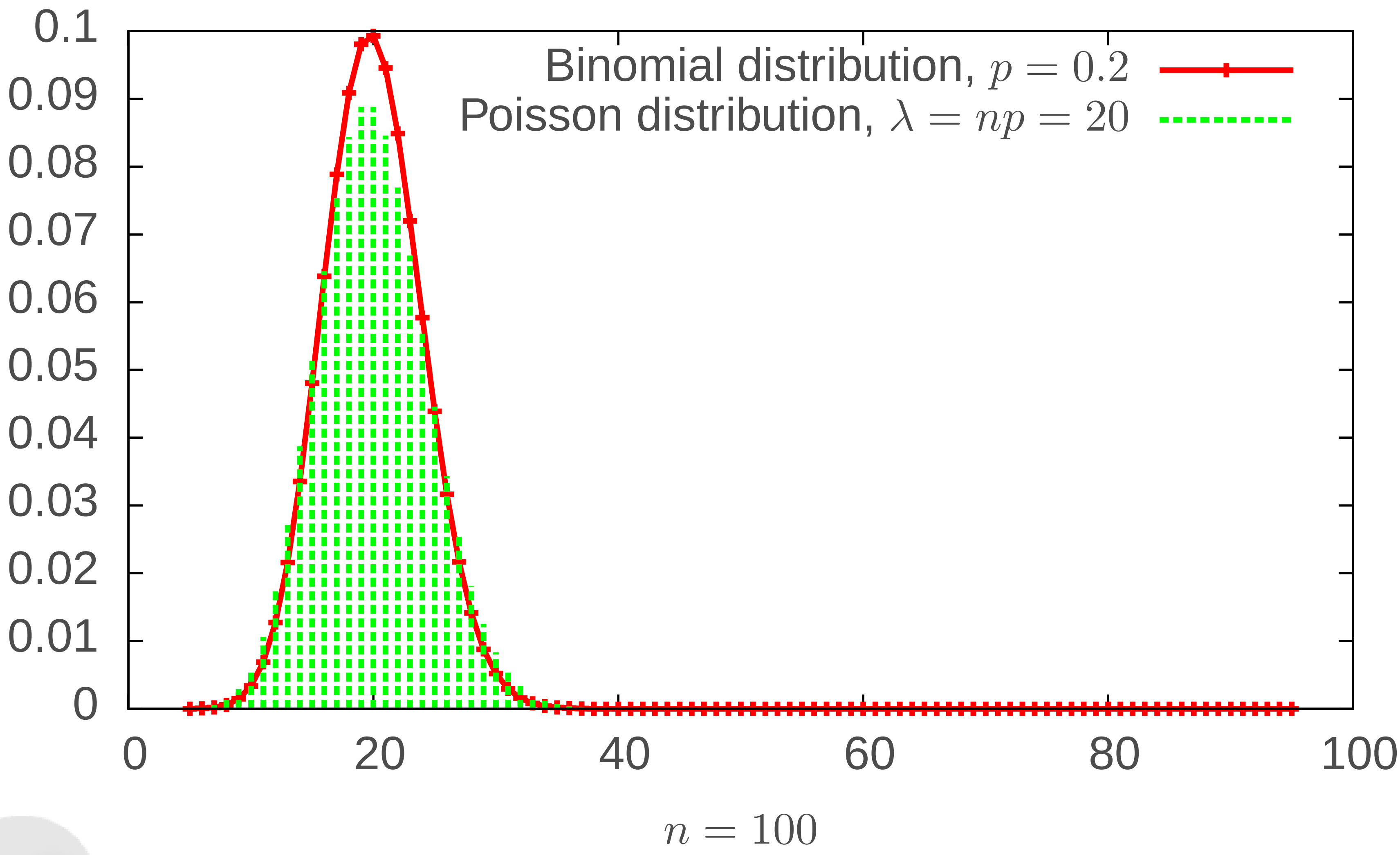
- Number of successes in a large number n of trials (binomial distribution)
- when the probability p of a success is small.
- Denote $\lambda = np$.

$$\Pr(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

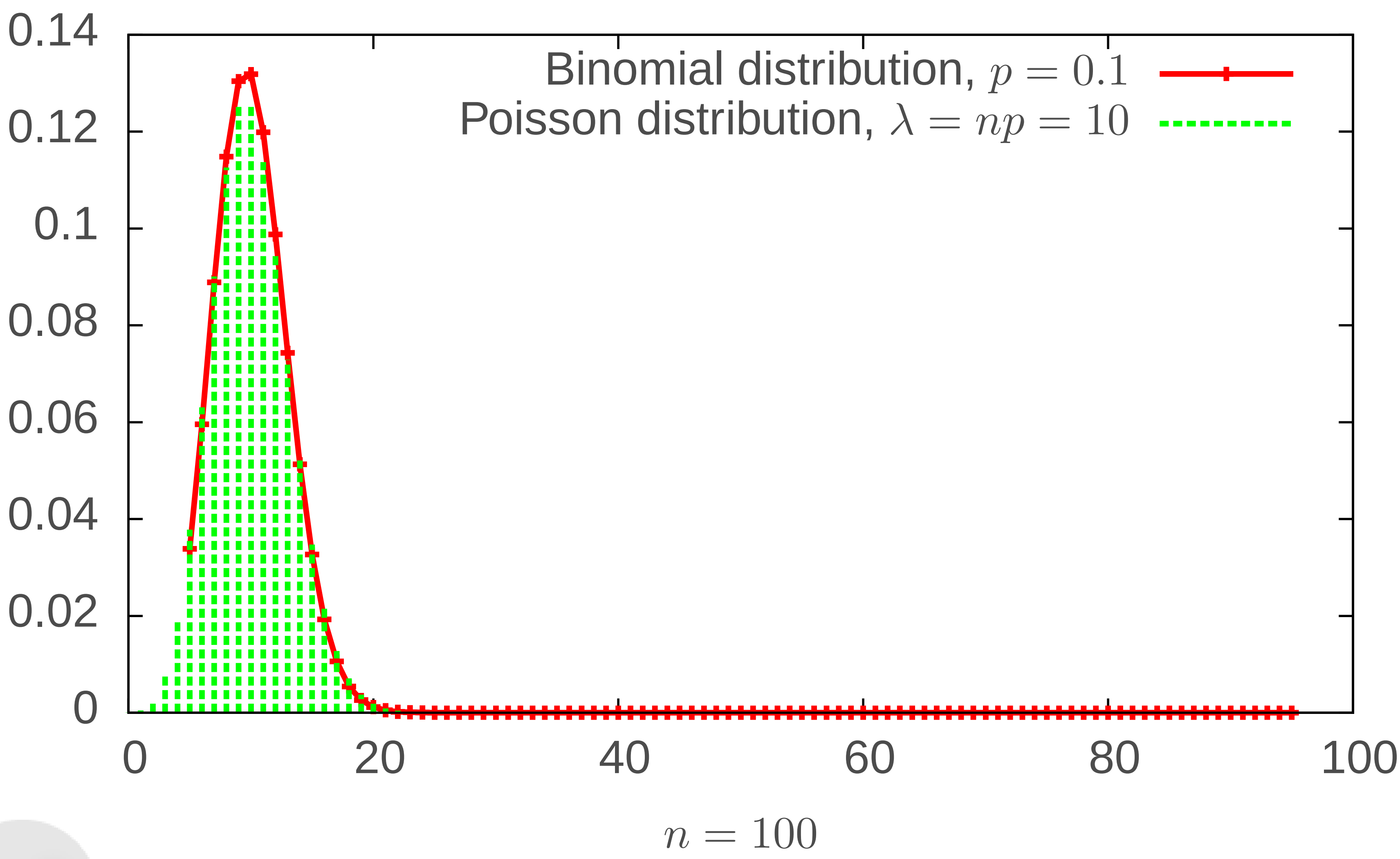
Property:

$$\mathrm{E}[X] = \mathrm{Var}(X) = \lambda.$$

Poisson random variable



Poisson random variable



Poisson process

- Events are occurring at random time points
- $N(t)$ is the number of events during $[0, t]$
- They constitute a Poisson process with rate $\lambda > 0$ if
  1. $N(0) = 0$,
  2. # of events occurring in disjoint time intervals are independent,
  3. distribution of $N(t+s) - N(t)$ depends on s, not on t,
  4. probability of one event in a small interval is approx. λh:

$$\lim_{h \to 0} \frac{\Pr(N(h) = 1)}{h} = \lambda,$$

  5. probability of two events in a small interval is approx. 0:

$$\lim_{h \to 0} \frac{\Pr(N(h) \geq 2)}{h} = 0.$$

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Poisson process

Property:

$$N(t) \sim \text{Poisson}(\lambda t), \quad \Pr(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

Inter-arrival times:

- S_k is the time when the kth event occurs,
- $X_k = S_k - S_{k-1}$ is the time elapsed between event $k-1$ and event k.
- $X_1 = S_1$
- Distribution of X_1: $\Pr(X_1 > t) = \Pr(N(t) = 0) = e^{-\lambda t}$.
- Distribution of X_2:

$$\begin{aligned}
\Pr(X_k > t | S_{k-1} = s) &= \Pr(0 \text{ events in }]s, s+t] | S_{k-1} = s) \\
&= \Pr(0 \text{ events in }]s, s+t]) \\
&= e^{-\lambda t}.
\end{aligned}$$

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Poisson process

- X_1 is an exponential random variable with mean $1/\lambda$
- X_2 is an exponential random variable with mean $1/\lambda$
- X_2 is independent of X_1.
- Same arguments can be used for $k = 3, 4 \ldots$.

Therefore, the CDF of X_k is, for any k,

$$F(t) = \Pr(X_k \leq t) = 1 - \Pr(X_k > t) = 1 - e^{-\lambda t}.$$

The pdf is

$$f(t) = \frac{dF(t)}{dt} = \lambda e^{-\lambda t}.$$

The inter-arrival times X_1, X_2,… are independent and identically distributed exponential random variables with parameter λ, and mean $1/\lambda$.

Poisson process

- Simulation of event times of a Poisson process with rate λ until time T:
  1. $t = 0$, $k = 0$.
  2. Draw $r \sim U(0,1)$.
  3. $t = t - \ln(r)/\lambda$.
  4. If $t > T$, STOP.
  5. $k = k + 1$, $S_k = t$.
  6. Go to step 2.

Non homogeneous Poisson process

- Assume that the rate varies with time, and call it $\lambda(t)$.
- The events constitute a non homogeneous Poisson process with rate $\lambda(t)$ if
  1. $N(0) = 0$
  2. # of events occurring in disjoint time intervals are independent,
  3. probability of one event in a small interval is approx. $\lambda(t)h$:

$$\lim_{h \to 0} \frac{\Pr\left((N(t+h) - N(t)) = 1\right)}{h} = \lambda(t),$$

  4. probability of two events in a small interval is approx. 0:

$$\lim_{h \to 0} \frac{\Pr\left((N(t+h) - N(t)) \geq 2\right)}{h} = 0.$$

Non homogeneous Poisson process

- Mean value function:

$$m(t) = \int_0^t \lambda(s)ds,\ t \geq 0.$$

- Poisson distribution:

$$N(t+s) - N(t) \sim \text{Poisson}(m(t+s) - m(t))$$

- Link with homogeneous Poisson process:
  - Consider a Poisson process with rate λ.
  - If an event occurs at time t, count it with probability $p(t)$.
  - The process of counted events is a non homogeneous Poisson process with rate $\lambda(t) = \lambda p(t)$.

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Non homogeneous Poisson process

Proof:

1. $N(0) = 0$ [OK]
2. # of events occurring in disjoint time intervals are independent, [OK]
3. probability of one event in a small interval is approx. $\lambda(t)h$: [?]

$$\lim_{h \to 0} \frac{\Pr\left((N(t+h) - N(t)) = 1\right)}{h} = \lambda(t),$$

4. probability of two events in a small interval is approx. 0: [OK]

$$\lim_{h \to 0} \frac{\Pr\left((N(t+h) - N(t)) \geq 2\right)}{h} = 0.$$

Non homogeneous Poisson process

- $N(t)$ number of events of the non homogeneous process
- $N'(t)$ number of events of the underlying homogeneous process

$$
\begin{aligned}
\Pr\left((N(t+h)-N(t))=1\right) &= \textstyle\sum_k \Pr\left((N'(t+h)-N'(t))=k, 1 \text{ is counted}\right) \\
&= \Pr\left((N'(t+h)-N'(t))=1, 1 \text{ is counted}\right) \\
&= \Pr\left((N'(t+h)-N'(t))=1\right)\Pr(1 \text{ is counted})
\end{aligned}
$$

$$
\begin{aligned}
\lim_{h\to 0}\frac{\Pr((N(t+h)-N(t))=1)}{h} &= \lim_{h\to 0}\frac{\Pr((N'(t+h)-N'(t))=1)}{h}\Pr(1 \text{ is counted}) \\
&= \lambda p(t).
\end{aligned}
$$

Non homogeneous Poisson process

Simulation of event times of a non homogeneous Poisson process with rate $\lambda(t)$ until time T:

1. Consider λ such that $\lambda(t) \leq \lambda$, for all $t \leq T$.
2. $t = 0$, $k = 0$.
3. Draw $r \sim U(0, 1)$.
4. $t = t - \ln(r)/\lambda$.
5. If $t > T$, STOP.
6. Generate $s \sim U(0, 1)$.
7. If $s \leq \lambda(t)/\lambda$, then $k = k + 1$, $S(k) = t$.
8. Go to step 3.

Summary

- Poisson random variable
- Poisson process
- Non homogeneous Poisson process
- Main assumption: events occur continuously and independently of one another
- Typical usage: arrivals of customers in a queue
- Easy to simulate