# Drawing from distributions

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# Discrete distributions

- Let $X$ be a discrete r.v. with pmf:

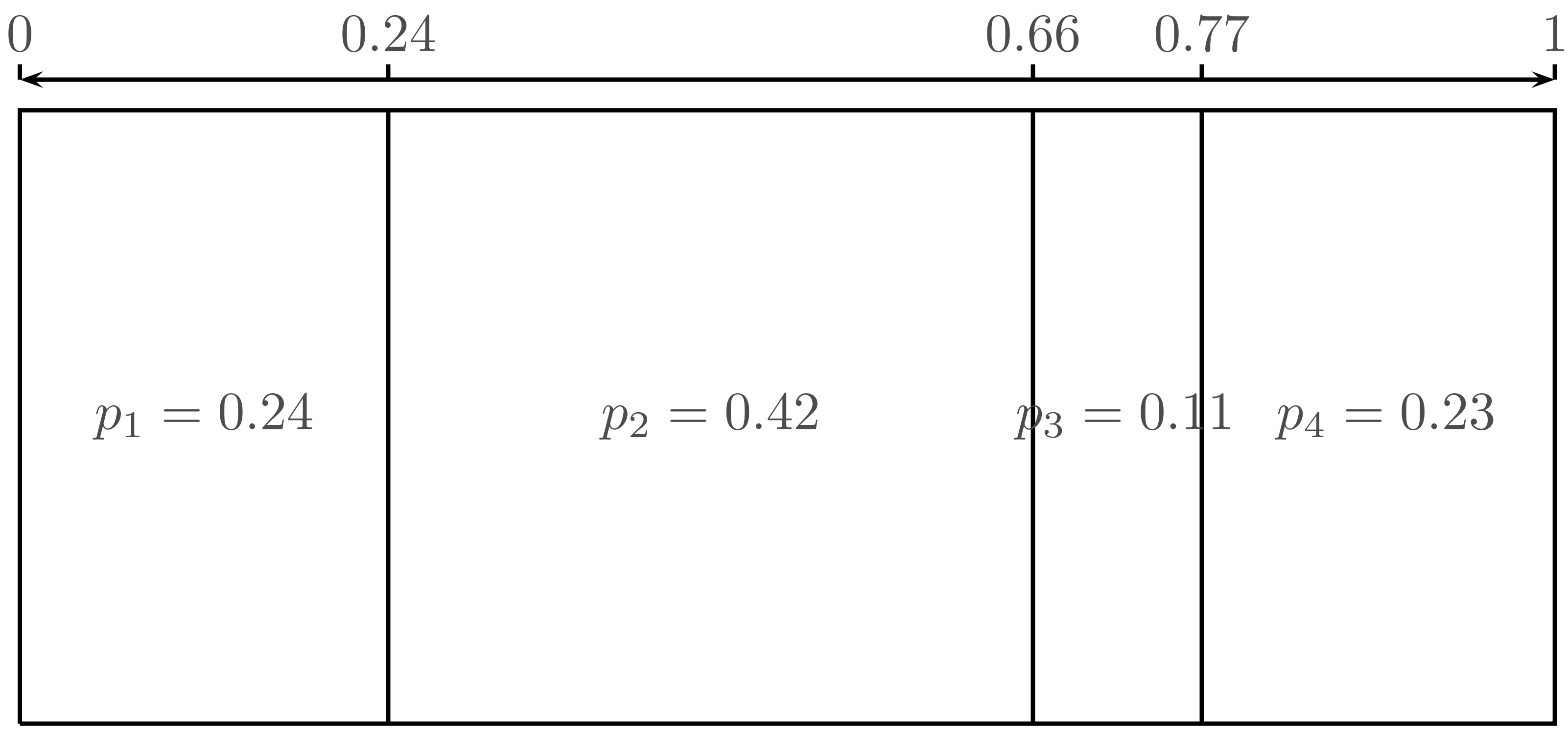
$$P(X = x_i) = p_i, \; i = 0, \ldots,$$

  where $\sum_i p_i = 1$.
- The support can be finite or infinite.
- The following algorithm generates draws from this distribution

1. Let $r$ be a draw from $U(0, 1)$.
2. Initialize $k = 0$, $p = 0$.
3. $p = p + p_k$.
4. If $r < p$, set $X = x_k$ and stop.
5. Otherwise, set $k = k + 1$ and go to step 3.

Inverse transform method

# Inverse Transform Method: illustration



# Discrete distributions

Acceptance-rejection technique

- Attributed to von Neumann.
- Mostly useful with continuous distributions.
- We want to draw from $X$ with pmf $p_i$.
- We know how to draw from $Y$ with pmf $q_i$.

Define a constant $c \geq 1$ such that

$$\frac{p_i}{q_i} \leq c \; \forall i \text{ s.t. } p_i > 0.$$

Algorithm:

1. `Draw` $y$ `from` $Y$
2. `Draw` $r$ `from` $U(0,1)$
3. `If` $r < \frac{p_y}{cq_y}$`, return` $x = y$ `and stop. Otherwise, start again.`

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# Acceptance-rejection: analysis

Probability to be accepted during a given iteration:

$$
\begin{aligned}
P(Y=y,\text{accepted}) &= P(Y=y) \quad P(\text{accepted}|Y=y) \\
&= q_y \qquad\qquad\quad p_y/cq_y \\
&= \frac{p_y}{c}
\end{aligned}
$$

Probability to be accepted:

$$
\begin{aligned}
P(\text{accepted}) &= \sum_y P(\text{accepted}|Y=y)P(Y=y) \\
&= \sum_y \frac{p_y}{cq_y} q_y \\
&= 1/c.
\end{aligned}
$$

Probability to draw $x$ at iteration $n$

$$
P(X=x|n) = (1-\tfrac{1}{c})^{n-1}\tfrac{p_x}{c}
$$

# Acceptance-rejection: analysis

Therefore,

$$
\begin{aligned}
P(X=x) &= \sum_{n=1}^{+\infty} P(X=x|n) \\
&= \sum_{n=1}^{+\infty} \left(1-\frac{1}{c}\right)^{n-1} \frac{p_x}{c} \\
&= c\frac{p_x}{c} \\
&= p_x.
\end{aligned}
$$

Reminder: geometric series

$$\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$$

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# Acceptance-rejection: analysis

Remarks:

- Average number of iterations: $c$
- The closer $c$ is to 1, the closer the pmf of $Y$ is to the pmf of $X$.

# Continuous distributions

## Inverse Transform Method

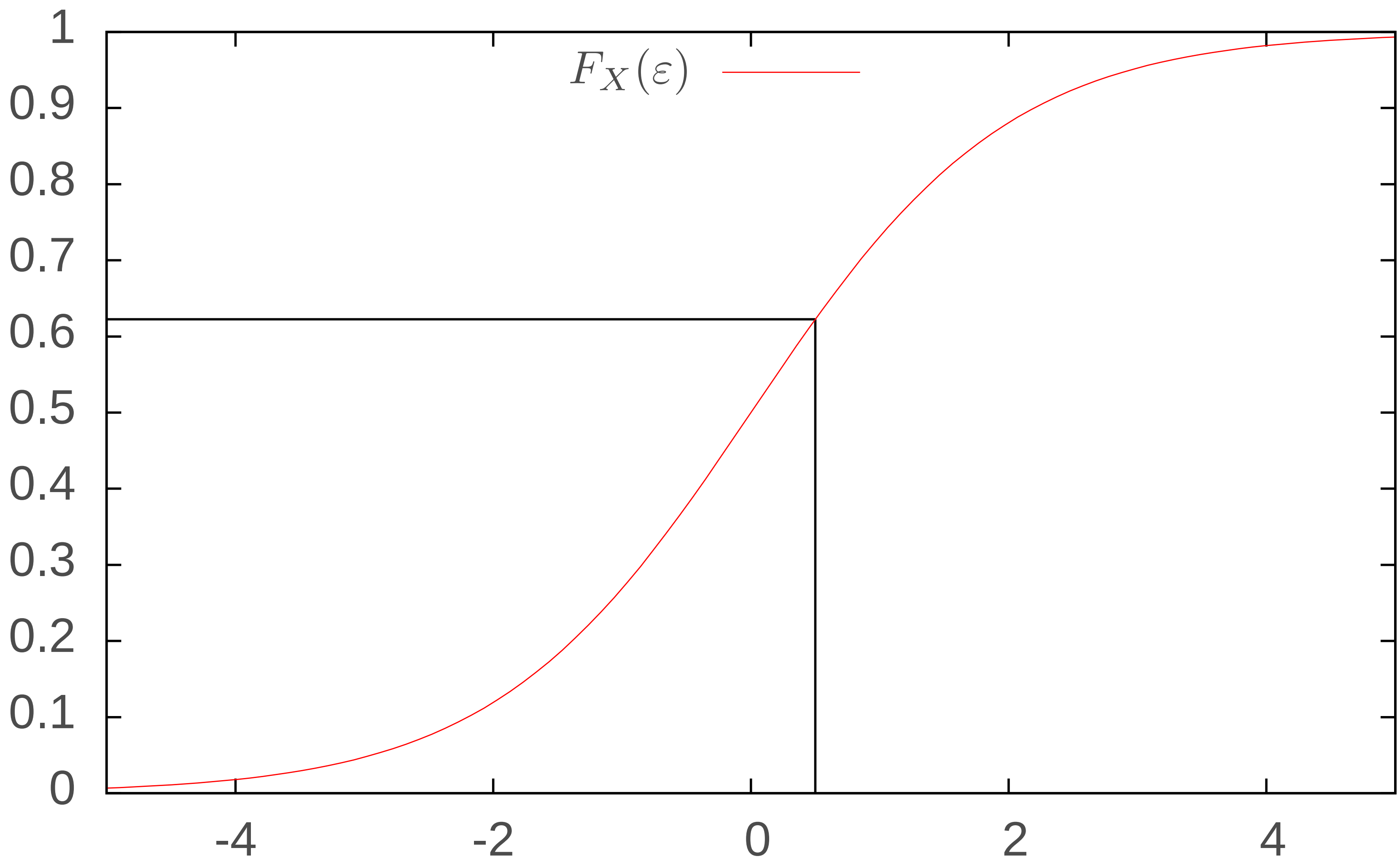
Idea:

- Let $X$ be a continuous r.v. with CDF $F_X(\varepsilon)$
- Draw $r$ from a uniform $U(0,1)$
- Generate $F_X^{-1}(r)$.

Motivation:

- $F_X$ is monotonically increasing
- It implies that $\varepsilon_1 \leq \varepsilon_2$ is equivalent to $F_X(\varepsilon_1) \leq F_X(\varepsilon_2)$.

# Inverse Transform Method



# Inverse Transform Method

More formally:

- Denote $F_U(\varepsilon) = \varepsilon$ the CDF of the r.v. $U(0,1)$
- Let $G$ be the distribution of the r.v. $F_X^{-1}(U)$

$$
\begin{aligned}
G(\varepsilon) &= \Pr(F_X^{-1}(U) \leq \varepsilon) \\
&= \Pr(F_X(F_X^{-1}(U)) \leq F_X(\varepsilon)) \\
&= \Pr(U \leq F_X(\varepsilon)) \\
&= F_U(F_X(\varepsilon)) \\
&= F_X(\varepsilon)
\end{aligned}
$$

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# Inverse Transform Method

Examples: let $r$ be a draw from $U(0,1)$

| Name | $F_X(\varepsilon)$ | Draw |
|---|---|---|
| Exponential($b$) | $1-e^{-\varepsilon/b}$ | $-b\ln r$ |
| Logistic($\mu$,$\sigma$) | $1/(1+\exp(-(\varepsilon-\mu)/\sigma))$ | $\mu-\sigma\ln(\frac{1}{r}-1)$ |
| Power($n$,$\sigma$) | $(\varepsilon/\sigma)^n$ | $\sigma r^{1/n}$ |

Note: the CDF is not always available (e.g. normal distribution).

# Continuous distributions

## Rejection Method

- We want to draw from $X$ with pdf $f_X$.
- We know how to draw from $Y$ with pdf $f_Y$.

Define a constant $c \geq 1$ such that

$$\frac{f_X(\varepsilon)}{f_Y(\varepsilon)} \leq c \; \forall \varepsilon$$

Algorithm:

1. Draw $y$ from $Y$
2. Draw $r$ from $U(0,1)$
3. If $r < \frac{f_X(y)}{cf_Y(y)}$, return $x = y$ and stop. Otherwise, start again.

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# Rejection Method: example

Draw from a normal distribution

- Let $\bar{X} \sim N(0,1)$ and $X = |\bar{X}|$
- Probability density function: $f_X(\varepsilon) = \frac{2}{\sqrt{2\pi}} e^{-\varepsilon^2/2}, \quad 0 < \varepsilon < +\infty$
- Consider an exponential r.v. with pdf $f_Y(\varepsilon) = e^{-\varepsilon}, \quad 0 < \varepsilon < +\infty$
- Then

$$\frac{f_X(\varepsilon)}{f_Y(\varepsilon)} = \frac{2}{\sqrt{2\pi}} e^{\varepsilon - \varepsilon^2/2}$$

- The ratio takes its maximum at $\varepsilon = 1$, therefore

$$\frac{f_X(\varepsilon)}{f_Y(\varepsilon)} \leq \frac{f_X(1)}{f_Y(1)} = \sqrt{2e/\pi} \approx 1.315.$$

- Rejection method, with $\frac{f_X(\varepsilon)}{cf_Y(\varepsilon)} = \frac{1}{\sqrt{e}} e^{\varepsilon - \varepsilon^2/2} = e^{\varepsilon - \frac{\varepsilon^2}{2} - \frac{1}{2}} = e^{-\frac{(\varepsilon-1)^2}{2}}$

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# Rejection Method: example

Draw from a normal

1. Draw $r$ from $U(0,1)$
2. Let $y = -\ln(1-r)$ (draw from the exponential)
3. Draw $s$ from $U(0,1)$
4. If $s < e^{-\frac{(y-1)^2}{2}}$ return $x = y$ and go to step 5. Otherwise, go to step 1.
5. Draw $t$ from $U(0,1)$.
6. If $t \leq 0.5$, return $x$. Otherwise, return $-x$.

Note: this procedure can be improved. See Ross, Chapter 5.

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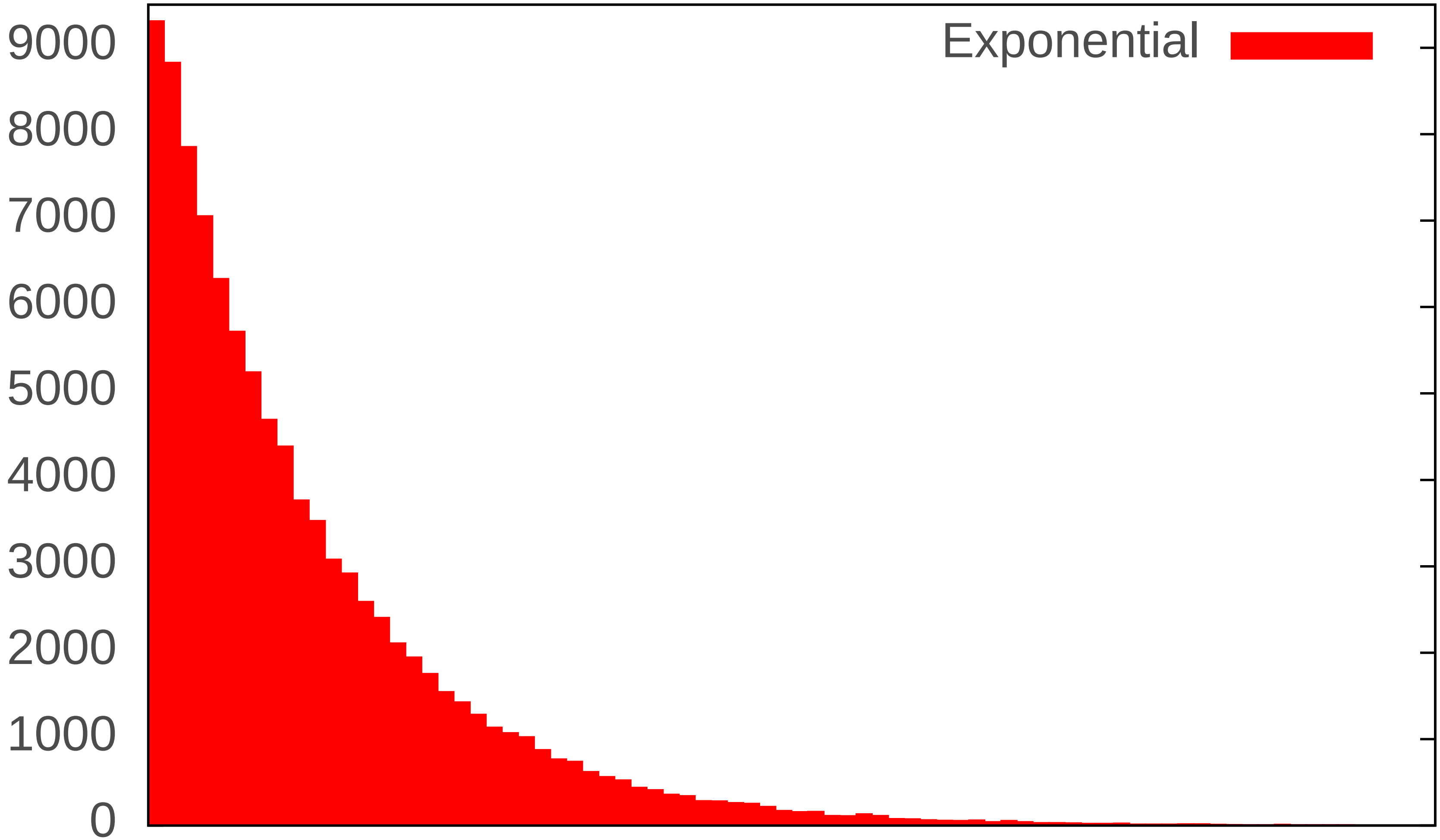
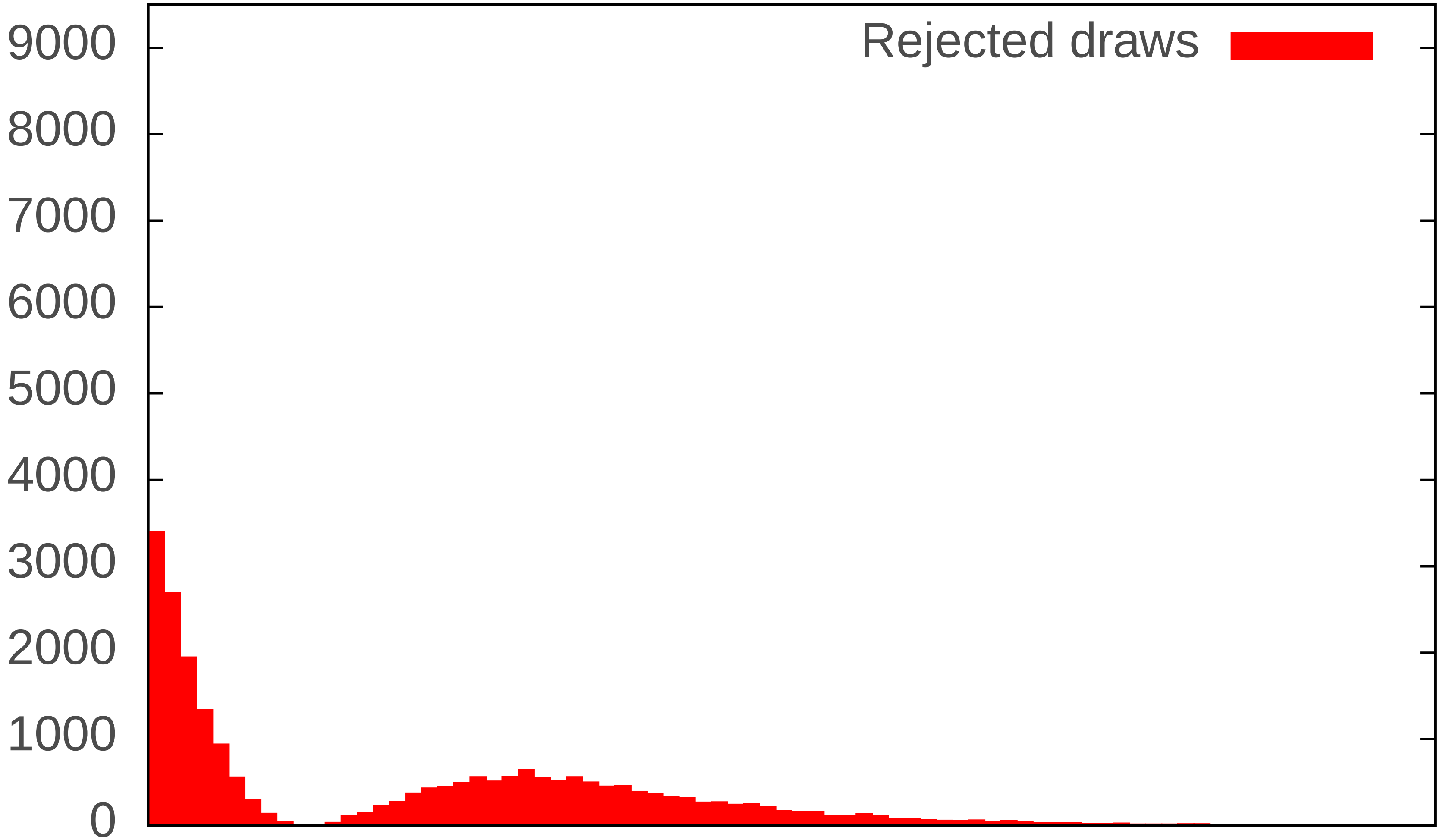
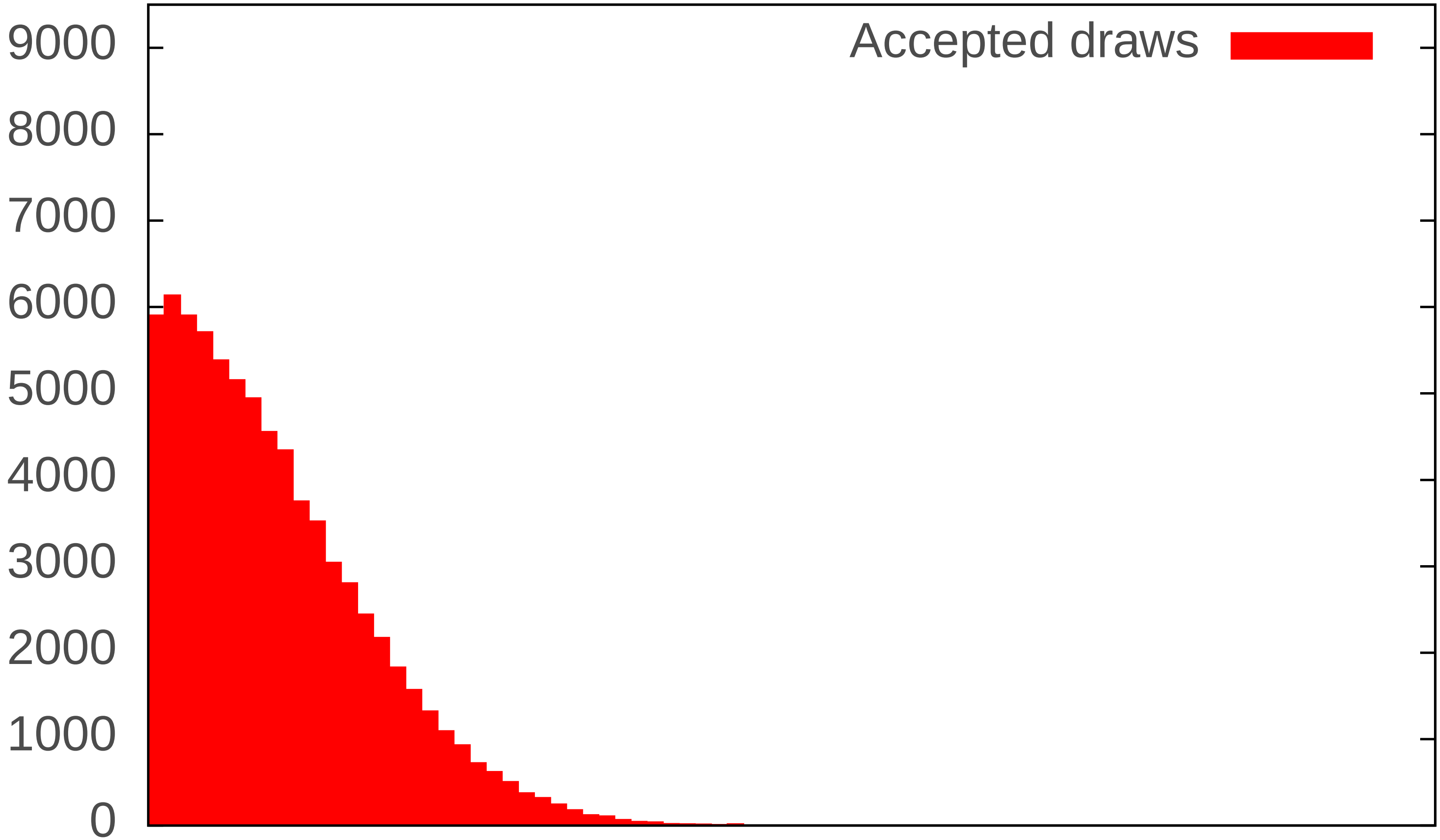
# Draws from the exponential



# Rejected draws



# Accepted draws

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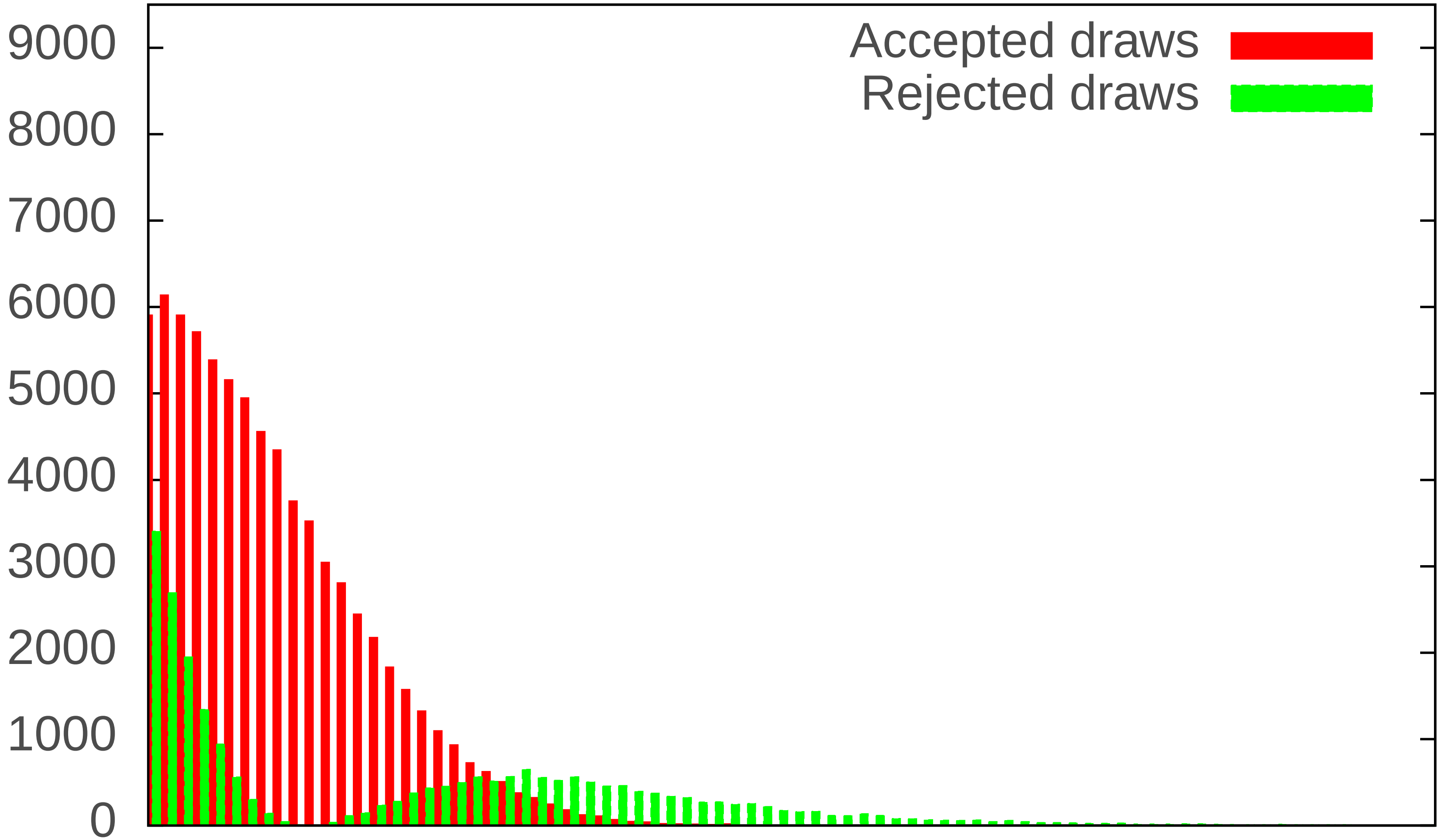
# Rejected and accepted draws



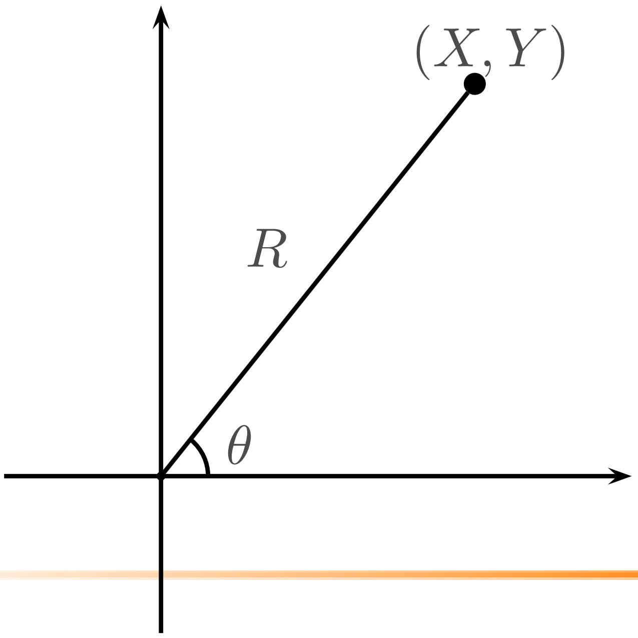
# The polar method

Draw from a normal distribution

- Let $X \sim N(0,1)$ and $Y \sim N(0,1)$ independent
- pdf:

$$f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = \frac{1}{2\pi} e^{-(x^2+y^2)/2}.$$

- Let $R$ and $\theta$ such that $R^2 = X^2 + Y^2$, and $\tan\theta = Y/X$.



# The polar method

Change of variables (reminder):

- Let $A$ be a multivariate r.v. distributed with pdf $f_A(a)$.
- Consider the change of variables $b = H(a)$ where $H$ is bijective and differentiable
- Then $B = H(A)$ is distributed with pdf

$$f_B(b) = f_A(H^{-1}(b)) \left| \det \left( \frac{dH^{-1}(b)}{db} \right) \right| .$$

Here: $A = (X, Y)$, $B = (R^2, \theta) = (T, \theta)$

$$H^{-1}(B) = \begin{pmatrix} T^{\frac{1}{2}} \cos\theta \\ T^{\frac{1}{2}} \sin\theta \end{pmatrix} \quad \frac{dH^{-1}(B)}{dB} = \begin{pmatrix} \frac{1}{2} T^{-\frac{1}{2}} \cos\theta & -T^{\frac{1}{2}} \sin\theta \\ \frac{1}{2} T^{-\frac{1}{2}} \sin\theta & T^{\frac{1}{2}} \cos\theta \end{pmatrix}$$

# The polar method

$$H^{-1}(b) = \begin{pmatrix} T^{\frac{1}{2}}\cos\theta \\ T^{\frac{1}{2}}\sin\theta \end{pmatrix} \quad \frac{dH^{-1}(b)}{db} = \begin{pmatrix} \frac{1}{2}T^{-\frac{1}{2}}\cos\theta & -T^{\frac{1}{2}}\sin\theta \\ \frac{1}{2}T^{-\frac{1}{2}}\sin\theta & T^{\frac{1}{2}}\cos\theta \end{pmatrix}$$

Therefore,

$$\left|\det\left(\frac{dH^{-1}(b)}{db}\right)\right| = \frac{1}{2}.$$

and

$$f_B(T,\theta) = \frac{1}{2}\frac{1}{2\pi}e^{-T/2}, \quad 0 < T < +\infty, \quad 0 < \theta < 2\pi.$$

Product of

- an exponential with mean 2: $\frac{1}{2}e^{-T/2}$
- a uniform on $[0, 2\pi[$: $1/2\pi$

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# The polar method

Therefore,

- $R^2$ and $\theta$ are independent
- $R^2$ is exponential with mean 2
- $\theta$ is uniform on $(0, 2\pi)$

Algorithm:

1. Let $r_1$ and $r_2$ be draws from $U(0, 1)$.
2. Let $R^2 = -2 \ln r_1$ (draw from exponential of mean 2)
3. Let $\theta = 2\pi r_2$ (draw from $U(0, 2\pi)$)
4. Let

$$\begin{aligned} X &= R\cos\theta &= \sqrt{-2\ln r_1}\cos(2\pi r_2) \\ Y &= R\sin\theta &= \sqrt{-2\ln r_1}\sin(2\pi r_2) \end{aligned}$$

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# The polar method

Issue: time consuming to compute sine and cosine
Solution: generate directly the sine and the cosine

- Draw a random point $(s_1, s_2)$ in the circle of radius one centered at $(0, 0)$.
- How? Draw a random point in the square $[-1, 1] \times [-1, 1]$ and reject points outside the circle
- Let $(R, \theta)$ be the polar coordinates of this point.
- $R^2 \sim U(0, 1)$ and $\theta \sim U(0, 2\pi)$ are independent

$$
\begin{aligned}
R^2 &= s_1^2 + s_2^2 \\
\cos\theta &= s_1/R \\
\sin\theta &= s_2/R
\end{aligned}
$$

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# The polar method

Original transformation:

$$\begin{aligned} X &= R\cos\theta = \sqrt{-2\ln r_1}\cos(2\pi r_2) \\ Y &= R\sin\theta = \sqrt{-2\ln r_1}\sin(2\pi r_2) \end{aligned}$$

Replace $r_1$ by $t = R^2 \sim U(0,1)$, and the sine and cosine as described above

$$\begin{aligned} t &= s_1^2 + s_2^2 \\ X &= R\cos\theta = \sqrt{-2\ln t}\frac{s_1}{\sqrt{t}} = s_1\sqrt{\frac{-2\ln t}{t}} \\ Y &= R\sin\theta = \sqrt{-2\ln t}\frac{s_2}{\sqrt{t}} = s_2\sqrt{\frac{-2\ln t}{t}} \end{aligned}$$

# The polar method

Algorithm:

1. Let $r_1$ and $r_2$ be draws from $U(0,1)$.
2. Define $s_1 = 2r_1 - 1$ and $s_2 = 2r_2 - 1$ (draws from $U(-1,1)$).
3. Define $t = s_1^2 + s_2^2$.
4. If $t > 1$, reject the draws and go to step 1.
5. Return

$$x = s_1\sqrt{\frac{-2\ln t}{t}} \text{ and } y = s_2\sqrt{\frac{-2\ln t}{t}}.$$

# Transformations of standard normal

- If $r$ is a draw from $N(0,1)$, then

$$s = br + a$$

  is a draw from $N(a, b^2)$

- If $r$ is a draw from $N(a, b^2)$, then

$$e^r$$

  is a draw from a log normal $LN(a, b^2)$ with mean

$$e^{a+(b^2/2)}$$

  and variance

$$e^{2a+b^2}(e^{b^2} - 1)$$

# Multivariate normal

- If $r_1$,...,$r_n$ are independent draws from $N(0,1)$, and

$$r = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$$

- then

$$s = a + Lr$$

is a vector of draws from the $n$-variate normal $N(a, LL^T)$, where

- $L$ is lower triangular, and
- $LL^T$ is the Cholesky factorization of the variance-covariance matrix

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# Multivariate normal

Example:

$$L = \begin{pmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix}$$

$$\begin{array}{rcl} s_1 & = & \ell_{11} r_1 \\ s_2 & = & \ell_{21} r_1 + \ell_{22} r_2 \\ s_3 & = & \ell_{31} r_1 + \ell_{32} r_2 + \ell_{33} r_3 \end{array}$$

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# Transforming draws

- Consider draws from the following distributions:
  - normal: $N(0,1)$ (draws denoted by $\xi$ below)
  - uniform: $U(0,1)$ (draws denoted by $r$ below)
- Draws $R$ from other distributions are obtained from nonlinear transforms.
- Lognormal(a,b)

$$f(x) = \frac{1}{xb\sqrt{2\pi}} \exp\left(\frac{-(\ln x - a)^2}{2b^2}\right) \qquad R = e^{a+b\xi}$$

# Transforming draws

- Cauchy(a,b)

$$f(x) = \left(\pi b \left(1 + \left(\frac{x-a}{b}\right)^2\right)\right)^{-1} \qquad R = a + b\tan\left(\pi(r - \frac{1}{2})\right)$$

- $\chi^2(a)$ ($a$ integer)

$$f(x) = \frac{x^{(a-2)/2}e^{-x/2}}{2^{a/2}\Gamma(a/2)} \qquad R = \sum_{j=1}^{a} \xi_j^2$$

- Erlang(a,b) ($b$ integer)

$$f(x) = \frac{(x/a)^{b-1}e^{-x/a}}{a(b-1)!} \qquad R = -a\sum_{j=1}^{b} \ln r_i$$

# Transforming draws

- Exponential(a)

$$F(x) = 1 - e^{-x/a} \qquad R = -a \ln r$$

- Extreme Value(a,b)

$$F(x) = 1 - \exp\left(-e^{-(x-a)/b}\right) \qquad R = a - b \ln(-\ln r)$$

- Logistic(a,b)

$$F(x) = \left(1 + e^{-(x-a)/b}\right)^{-1} \qquad R = a + b \ln\left(\frac{r}{1-r}\right)$$

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# Transforming draws

- Pareto(a,b)

$$F(x) = 1 - \left(\frac{a}{x}\right)^b \qquad R = a(1-r)^{-1/b}$$

- Standard symmetrical triangular distribution

$$f(x) = \begin{cases} 4x & \text{if } 0 \leq x \leq 1/2 \\ 4(1-x) & \text{if } 1/2 \leq x \leq 1 \end{cases} \qquad R = \frac{r_1 + r_2}{2}$$

- Weibull(a,b)

$$F(x) = 1 - e^{-\left(\frac{x}{a}\right)^b} \quad R = a(-\ln r)^{1/b}$$

# Appendix

Uniform distribution: $X \sim U(a,b)$

- pdf

$$f_X(x) = \begin{cases} 1/(b-a) & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

- CDF

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq a, \\ (x-a)/(b-a) & \text{if } a \leq x \leq b, \\ 1 & \text{if } x \geq b. \end{cases}$$

- Mean, median: $(a+b)/2$.
- Variance: $(b-a)^2/12$

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