Introduction to simulation

Michel Bierlaire

michel.bierlaire@epfl.ch

Transport and Mobility Laboratory

Modeling

- A system can be seen as a black box, modeled by

$$y = h(x, u)$$

- Example: a car
- x captures the state of the system (e.g. speed, position of other vehicles)
- u captures possible human controls on the system (e.g. acceleration/deceleration)
- y represents indicators of performance (e.g. oil consumption).

Modeling

- The model f is usually decomposed to reflect the interactions of the subsystems
- For example,
  - a car-following model captures the target speed of the driver,
  - an engine model derives the actual consumption as a function of the acceleration.
- In practice, such a model is never representing accurately the reality.

Modeling

- Uncertainty is captured by random variables

$$Y = h(X, U, \varepsilon)$$

  where X, U, ε and Y are random variables.
- We are interested in the distribution of Y.
- When f is complex (that is, a combination of many models),
  - the distribution of Y is complex,
  - even if the distributions of X, U and ε are simple.

Modeling

- Assume for the sake of simplicity that all r.v. are continuous
- Denote Z the random vector (X, U, ε)
- Distribution of Y:

$$f_Y(y) = \int_z f_Y(y|Z = z) f_Z(z) dz.$$

- In general, no analytical formula is available for f_Y
- When the dimension of Z is large, numerical integration is not an option.
- Solution: Monte-Carlo integration

Monte-Carlo integration

Compute

$$I = \int_0^1 f(x)dx$$

If $\varepsilon \sim U(0,1)$, then

$$I = E[f(\varepsilon)]$$

If r_1,..., r_k are k independent draws from ε, then

$$I = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} f(r_i).$$

Requires only to evaluate f.

Monte-Carlo integration

Intuitive example: estimation of π

Monte-Carlo integration

$$f_Y(y) = \int_z f_Y(y|Z=z) f_Z(z) dz.$$

- Idea of simulation:
  - Draw R realizations of Z: $z_1, \ldots, z_R$.
  - Approximate:

$$f_Y(y) = \int_z f_Y(y|Z=z) f_Z(z) dz \approx \frac{1}{R} \sum_{r=1}^{R} 1(h(z_r) = y).$$

Challenges

- How to generate draws from Z?
- How to represent complex systems? (specification of h)
- How large R should be?
- How good is the approximation of the integral?

TRANSP-OR

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Pseudo-random numbers

- Deterministic sequence of numbers
- which have the appearance of draws from a $U(0,1)$ distribution

Typical sequence:

$$x_n = ax_{n-1} \text{ modulo } m$$

- This has a period of the order of m
- So, m should be a large prime number
- For instance: $m = 2^{31} - 1$ and $a = 7^5$
- x_n/m lies in the $[0, 1[$ interval

Outline

- Drawing from distributions
- Discrete event simulation
- Data analysis
- Variance reduction
- Markov Chain Monte Carlo