Decomposition for Network Design

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February 25, 2016

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Outline of lesson 3: Introduction to decomposition methods

- Cutting-plane method
- Column generation
- Column-and-row generation
- Lagrangian relaxation
- Benders decomposition

Generic MIP model

$$
\begin{aligned}
Z(M) = \min\ & cx + fy \\
& Ax = b \\
& Bx + Dy \geq e \\
& Gy \geq h \\
& x, y \geq 0 \\
& y \text{ integer}
\end{aligned}
$$

- Hypotheses: M is feasible and bounded

Cutting-plane method

- Assume that there are "too many" constraints in $Bx + Dy \geq e$
- Generate a small subset of these
- Solve the LP relaxation to obtain an optimal solution $(\overline{x}, \overline{y})$
- Solve the **separation problem**:
  1) determine that no more valid inequalities among $Bx + Dy \geq e$ are violated by $(\overline{x}, \overline{y})$
  2) otherwise, find at least one valid inequality among $Bx + Dy \geq e$ that cuts $(\overline{x}, \overline{y})$
- Case 1): stop the algorithm
  Case 2): add the valid inequalities to the model
- Repeat this iterative process until no more cuts are identified
- This method solves $LP(M)$, but not M! What should we do?

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- Branch-and-**cut**: cutting-plane at every node of the B&B tree

Column generation method

- Assume that there are "too many" variables in x
- Generate a small subset of these (just enough to define a basic feasible solution to $LP(M)$)
- Solve the LP relaxation to obtain an optimal solution $(\overline{x}, \overline{y})$
- Solve the **pricing problem**:
  1) determine that no more x variables have a negative reduced cost
  2) otherwise, find at least one x variable with a negative reduced cost
- Case 1): stop the algorithm
  Case 2): add the columns to the model
- Repeat this iterative process until no more variables are found
- This method solves $LP(M)$, but not M! What should we do?

Column generation method

- Assume that there are "too many" variables in x
- Generate a small subset of these (just enough to define a basic feasible solution to $LP(M)$)
- Solve the LP relaxation to obtain an optimal solution $(\bar{x}, \bar{y})$
- Solve the **pricing problem**:
  1) determine that no more x variables have a negative reduced cost
  2) otherwise, find at least one x variable with a negative reduced cost
- Case 1): stop the algorithm
  Case 2): add the columns to the model
- Repeat this iterative process until no more variables are found
- This method solves $LP(M)$, but not M! What should we do?
- Branch-and-**price**: column generation at every node of the B&B tree

Column-and-row generation method

- Assume that there are "too many" variables in x *and* "too many" constraints in $Bx + Dy \geq e$
- Generate subsets of the x variables and of the constraints in $Bx + Dy \geq e$
- Solve the LP relaxation to obtain an optimal solution $(\overline{x}, \overline{y})$
- Solve the pricing problem to generate new columns
- Solve the separation problem to obtain new cuts
- Repeat this iterative process until no more columns or no more cuts are found
- Branch-and-**price-and-cut**: column-and-row generation at every node of the B&B tree

Lagrangian relaxation

- $Bx + Dy \geq e$ can be seen as "complicating" constraints
- Relax them, but instead of dropping them, add them to the objective by associating **Lagrange multipliers** $\alpha \geq 0$ representing penalties associated to their violation

$$
\begin{aligned}
Z(LR(\alpha)) = \min cx + fy + \alpha(e - Bx - Dy) \\
Ax = b \\
Gy \geq h \\
x, y \geq 0 \\
y \text{ integer}
\end{aligned}
$$

- For any $\alpha \geq 0$, we have $Z(LR(\alpha)) \leq Z(M)$
- Prove this property!

Lagrangian subproblem

$$Z(LR(\alpha)) = \alpha e + Z_x(\alpha) + Z_y(\alpha)$$

$$Z_x(\alpha) = \min(c - \alpha B)x$$
$$Ax = b$$
$$x \geq 0$$

$$Z_y(\alpha) = \min(f - \alpha D)y$$
$$Gy \geq h$$
$$y \geq 0$$
$$y \text{ integer}$$

- Hypothesis: both problems are bounded

Lagrangian dual

- The best values for the Lagrange multipliers are obtained by solving the **Lagrangian dual**:

$$Z(LD(M)) = \max_{\alpha \geq 0} Z(LR(\alpha))$$

- Let $Y = \{y \geq 0 \text{ and integer} | Gy \geq h\}$ and recall that $conv(Y)$ is a polyhedron
- **Primal interpretation of Lagrangian duality**:

$$Z(LD(M)) = \min cx + fy$$

$$Ax = b$$

$$x \geq 0$$

$$Bx + Dy \geq e$$

$$y \in conv(Y)$$

- Show this result by using Minkowski's theorem and LP duality

Lagrangian dual and LP relaxation

- We know $conv(Y) \subseteq \overline{Y}$, where $\overline{Y} = \{y \geq 0 | Gy \geq h\}$
- The primal interpretation of Lagrangian duality implies $Z(LD(M)) \geq Z(LP(M))$
- If the Lagrangian subproblem has the integrality property ($conv(Y) = \overline{Y}$), then $Z(LD(M)) = Z(LP(M))$
- Even when this is the case, maybe there is another Lagrangian relaxation such that $Z(LD(M)) \geq Z(LP(M))$
- Write down the Lagrangian subproblem and the Lagrangian dual if one relaxes the constraints $Ax = b$

Solving the Lagrangian dual by column generation

- Solve the **master problem** derived from the primal interpretation of Lagrangian duality and Minkowski's theorem:

$$Z(LD(M)) = \min cx + \sum_{k \in K} \lambda_k (f y_k)$$

$$Ax = b$$

$$Bx + \sum_{k \in K} \lambda_k (D y_k) \geq e$$

$$\sum_{k \in K} \lambda_k = 1$$

$$x \geq 0,\ \lambda_k \geq 0,\ k \in K$$

- Use column generation for this LP with "too many" variables
- The pricing problem is simply the Lagrangian subproblem

Solving the Lagrangian dual by subgradient optimization

- $Z(LR(\alpha))$ is continuous and concave (nice), but non-differentiable (ugly) function of α
- A **subgradient** of $Z(LR(\alpha))$ at $\overline{\alpha}$ is given by $(e - B\overline{x} - D\overline{y})$, where $(\overline{x}, \overline{y})$ solves the Lagrangian subproblem for $\alpha = \overline{\alpha}$
- At each iteration, a subgradient method finds a new α by taking a step in the direction of a subgradient
- Although there are convergence results, this is essentially a heuristic method that is very quick at each iteration
- Suggestion: use subgradient optimization at the beginning to get "good" Lagrange multipliers, then switch to column generation to get the best possible lower bound

Benders decomposition

- y can be seen as "complicating" variables
- Fix them at feasible values $\overline{y}$, i.e., such that $\overline{y} \in Y = \{y \geq 0 \text{ and integer} | Gy \geq h\}$
- We then have to solve the following LP, called **Benders subproblem**:

$$Z_x(\overline{y}) = \min cx$$

$$Ax = b$$

$$Bx \geq e - D\overline{y}$$

$$x \geq 0$$

- This problem is bounded (since M is bounded), but not necessarily feasible

Dual of Benders subproblem

$$Z_x(\overline{y}) = \max \pi b + \alpha(e - D\overline{y})$$
$$\pi A + \alpha B \leq c$$
$$\alpha \geq 0$$

- Let $D = \{(\pi, \alpha) | \pi A + \alpha B \leq c,\ \alpha \geq 0\}$ be the **dual polyhedron**
- If the Benders subproblem is infeasible, then its dual is unbounded and there exists an extreme ray of D, (π_j, α_j), such that $\pi_j b + \alpha_j(e - D\overline{y}) > 0$
- If the Benders subproblem is feasible, then its dual is also feasible and there exists and extreme point of D, (π_k, α_k), such that $\pi_k b + \alpha_k(e - D\overline{y}) = Z_x(\overline{y})$

Benders reformulation

$$Z(M) = \min_{y \in Y} fy + Z_x(y)$$

$$Z_x(y) = \{\max_{k \in K} (\pi_k b + \alpha_k (e - Dy)) \,|\, \pi_j b + \alpha_j (e - Dy) \leq 0,\ j \in J\}$$

- We can replace $Z_x(y)$ by a variable z and obtain the following MIP:

$$Z(M) = \min_{y \in Y, z} fy + z$$

$$\pi_k b + \alpha_k (e - Dy)) \leq z, \quad k \in K$$

$$\pi_j b + \alpha_j (e - Dy) \leq 0, \quad j \in J$$

Solving the Benders reformulation

- We can solve this **master problem** with "too many" constraints by a cutting-plane method
- Start by solving a relaxed master problem with no constraints associated to K and J
- At every iteration, solve the Benders subproblem to derive
  1) an extreme point (π_k, α_k) or
  2) an extreme ray (π_j, α_j)
- Case 1): add the **optimality cut** $\pi_k b + \alpha_k(e - Dy)) \leq z$
  Case 2): add the **feasibility cut** $\pi_j b + \alpha_j(e - Dy) \leq 0$
- At every iteration, let $(\overline{y}, \overline{z})$ be the optimal solution to the relaxed master problem and $\overline{x}$ be the optimal solution to the Benders subproblem (if it is feasible)
- $f\overline{y} + \overline{z} \leq Z(M) \leq f\overline{y} + c\overline{x}$
- The algorithm stops when $\overline{z} = c\overline{x}$