Decomposition for Network Design

Bernard Gendron*

February 17-18, 2016

EPFL, Lausanne, Switzerland

* CIRRELT and Département d'informatique et de recherche opérationnelle, Université de Montréal, Canada

Outline of lesson 1: Introduction

Contents of the course

Basic notions of LP and MIP

Basic notions of polyhedral theory

Modeling alternatives in MIP

Contents of the course

- Network design: many applications in telecommunications, transportation and logistics
- Multicommodity capacitated network design:
  - Interplay between investment and operational costs
  - Multiple commodities
  - Capacity constraints
- Decomposition methods from mathematical programming:
  - Lagrangian relaxation
  - Benders decomposition
  - Cutting-plane method
  - Column generation method
  - Colum-and-row generation method
  - Branch-and-X

Generic MIP model

$$Z(M) = \min cx + fy$$
$$Ax = b$$
$$Bx + Dy \geq e$$
$$Gy \geq h$$
$$x, y \geq 0$$
$$y \text{ integer}$$

- Hypotheses: M is feasible and bounded

LP relaxation

$$Z(LP(M)) = \min cx + fy$$
$$Ax = b$$
$$Bx + Dy \geq e$$
$$Gy \geq h$$
$$x, y \geq 0$$

- How to solve $LP(M)$?

LP relaxation

$$Z(LP(M)) = \min cx + fy$$
$$Ax = b$$
$$Bx + Dy \geq e$$
$$Gy \geq h$$
$$x, y \geq 0$$

- How to solve $LP(M)$?
- (Primal) Simplex method: give algebraic and geometric interpretations

LP relaxation

$$Z(LP(M)) = \min cx + fy$$
$$Ax = b$$
$$Bx + Dy \geq e$$
$$Gy \geq h$$
$$x, y \geq 0$$

- How to solve $LP(M)$?
- (Primal) Simplex method: give algebraic and geometric interpretations
- Other methods?

LP relaxation dual

$$
\begin{aligned}
Z(LP(M)) &= \min cx + fy \\
Ax &= b \quad (\pi) \\
Bx + Dy &\geq e \quad (\alpha \geq 0) \\
Gy &\geq h \quad (\theta \geq 0) \\
x, y &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
Z(LPD(M)) &= \max \pi b + \alpha e + \theta h \\
\pi A + \alpha B &\leq c \quad (x \geq 0) \\
\alpha D + \theta G &\leq f \quad (y \geq 0) \\
\alpha, \theta &\geq 0
\end{aligned}
$$

- **Weak duality**: $\pi b + \alpha e + \theta h \leq cx + fy$
  for any pair of primal-dual **feasible** solutions (x, y)-(π, α, θ)

LP relaxation dual

$$
\begin{aligned}
Z(LP(M)) = \min\ & cx + fy \\
Ax &= b \quad (\pi) \\
Bx + Dy &\geq e \quad (\alpha \geq 0) \\
Gy &\geq h \quad (\theta \geq 0) \\
x, y &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
Z(LPD(M)) = \max\ & \pi b + \alpha e + \theta h \\
\pi A + \alpha B &\leq c \quad (x \geq 0) \\
\alpha D + \theta G &\leq f \quad (y \geq 0) \\
\alpha, \theta &\geq 0
\end{aligned}
$$

- **Weak duality**: $\pi b + \alpha e + \theta h \leq cx + fy$
  for any pair of primal-dual **feasible** solutions (x, y)-(π, α, θ)
- **Strong duality**: $\pi b + \alpha e + \theta h = cx + fy$
  for any pair of primal-dual **optimal** solutions (x, y)-(π, α, θ)

LP relaxation dual

$$Z(LP(M)) = \min cx + fy$$
$$Ax = b \quad (\pi)$$
$$Bx + Dy \geq e \quad (\alpha \geq 0)$$
$$Gy \geq h \quad (\theta \geq 0)$$
$$x, y \geq 0$$

$$Z(LPD(M)) = \max \pi b + \alpha e + \theta h$$
$$\pi A + \alpha B \leq c \quad (x \geq 0)$$
$$\alpha D + \theta G \leq f \quad (y \geq 0)$$
$$\alpha, \theta \geq 0$$

- **Weak duality**: $\pi b + \alpha e + \theta h \leq cx + fy$
  for any pair of primal-dual **feasible** solutions (x, y)-(π, α, θ)
- **Strong duality**: $\pi b + \alpha e + \theta h = cx + fy$
  for any pair of primal-dual **optimal** solutions (x, y)-(π, α, θ)
- **Complementary slackness**: $\alpha(Bx + Dy - e) = 0$,
  $\theta(Gy - h) = 0$, $(c - \pi A - \alpha B)x = 0$, $(f - \alpha D - \theta G)y = 0$
  for any pair of primal-dual **optimal** solutions (x, y)-(π, α, θ)

Branch-and-bound (B&B) algorithm

- To solve M, we can use B&B: how does it work?

Branch-and-bound (B&B) algorithm

- To solve M, we can use B&B: how does it work?
- Main features of B&B:
  - Lower (dual) bounds: $LP(M)$ ($LPD(M)$) with dual (primal) simplex
  - Upper (primal) bounds: LP-based heuristics
  - Pruning (fathoming): what are the rules to prune a node?

Branch-and-bound (B&B) algorithm

- To solve M, we can use B&B: how does it work?
- Main features of B&B:
  - Lower (dual) bounds: $LP(M)$ ($LPD(M)$) with dual (primal) simplex
  - Upper (primal) bounds: LP-based heuristics
  - Pruning (fathoming): what are the rules to prune a node?
  - Branching: how does it work?

Branch-and-bound (B&B) algorithm

- To solve M, we can use B&B: how does it work?
- Main features of B&B:
  - Lower (dual) bounds: $LP(M)$ ($LPD(M)$) with dual (primal) simplex
  - Upper (primal) bounds: LP-based heuristics
  - Pruning (fathoming): what are the rules to prune a node?
  - Branching: how does it work?
  - Selection (search): depth-first or best-first or other rules?

Branch-and-bound (B&B) algorithm

- To solve M, we can use B&B: how does it work?
- Main features of B&B:
  - Lower (dual) bounds: $LP(M)$ ($LPD(M)$) with dual (primal) simplex
  - Upper (primal) bounds: LP-based heuristics
  - Pruning (fathoming): what are the rules to prune a node?
  - Branching: how does it work?
  - Selection (search): depth-first or best-first or other rules?
  - Other ingredients: preprocessing, reduced-cost domain reduction
- Atamturk A., Savelsbergh M.W.P., "Integer Programming Software Systems", *Annals of Operations Research* 140, 67-124, 2005

Polyhedral theory: basic definitions

- **Polyhedron**: set of solutions to a finite set of linear inequalities
- **Affine independence**: $x_0, x_1, \ldots, x_k$ are affinely independent iff $\omega_i = 0$, $i = 0, 1, \ldots k$, is the unique solution of the system $\sum_{i=0}^{k} \omega_i x_i = 0$, $\sum_{i=0}^{k} \omega_i = 0$
- **Dimension of polyhedron** P: maximum number of affinely independent points in a polyhedron - 1, noted $dim(P)$
- **Valid inequality**: an inequality is valid for a set S if it is satisfied by all points in the set
- **Face of polyhedron** P: a face is the set of points in P that satisfies a valid inequality for P at equality
- **Representation of face** F: the valid inequality for polyhedron P that is satisfied at equality by F is said to represent face F
- **Proper face of polyhedron** P: any face F of P such that $F \neq \emptyset, P$ (which is itself a polyhedron with $dim(F) < dim(P)$)

Facet description of polyhedra

- **Description of polyhedron** P: any set of linear inequalities whose solutions define P (there is an infinite number of descriptions for any polyhedron!)
- **Facet of polyhedron** P: proper face F of P such that $dim(F) = dim(P) - 1$
- **Description of polyhedron** P **by its facets**:
  - If F is a facet of P, then any description of P must include at least one inequality representing F (called **facet-defining inequality**)
  - Every inequality that represents a face that is not a facet is not necessary in the description of P

Extreme point description of polyhedra

- Hypothesis: all variables used to describe polyhedron P are nonnegative
- **Extreme point of polyhedron** P: a point of P that cannot be written as a convex combination of two other points of P
- Do you know any other characterization of an extreme point?

Extreme point description of polyhedra

- Hypothesis: all variables used to describe polyhedron P are nonnegative
- **Extreme point of polyhedron** P: a point of P that cannot be written as a convex combination of two other points of P
- Do you know any other characterization of an extreme point?
- **Cone** P^0 **associated to polyhedron** P: obtained from the description of P by replacing the right-hand side by 0
- **Ray of** P: vector belonging to P^0, the cone associated to P
- **Extreme ray of** P: ray of P that cannot be written as nonnegative combination of two other rays that do not point in the same direction
- **Minkowski's theorem**: every point of P can be written as a convex combination of the extreme points of P plus a nonnegative combination of the extreme rays of P
- Make sure you understand this important result! Write it down in mathematical form. What is happening if P is a bounded polyhedron?

Convexity and polyhedra

- **Convex hull of set** S: set of points that can be expressed as convex combination of the points of S, denoted $conv(S)$
- $conv(S) = S$, if S is convex, so $conv(P) = P$ if P is a polyhedron!
- If S is the feasible region of the MIP: $\min_{x \in S} cx$, then $conv(S)$ is a polyhedron
- This implies that, by replacing the feasible region S by $conv(S)$, we obtain the LP: $\min_{x \in conv(S)} cx$

Relationships between LP and MIP

- Let $\overline{S}$ be the feasible region of the LP relaxation: $\min_{x \in \overline{S}} cx \leq \min_{x \in S} cx$
- Since $conv(S)$ is contained in any convex set that includes S, we have $conv(S) \subseteq \overline{S}$, therefore $\min_{x \in \overline{S}} cx \leq \min_{x \in conv(S)} cx$
- We can even show that, if the MIP is feasible and bounded, then $\min_{x \in S} cx = \min_{x \in conv(S)} cx$
- So, a MIP is just an LP! What is the problem, then?

Relationships between LP and MIP

- Let $\overline{S}$ be the feasible region of the LP relaxation: $\min_{x \in \overline{S}} cx \leq \min_{x \in S} cx$
- Since $conv(S)$ is contained in any convex set that includes S, we have $conv(S) \subseteq \overline{S}$, therefore $\min_{x \in \overline{S}} cx \leq \min_{x \in conv(S)} cx$
- We can even show that, if the MIP is feasible and bounded, then $\min_{x \in S} cx = \min_{x \in conv(S)} cx$
- So, a MIP is just an LP! What is the problem, then?
- In general, a description of $conv(S)$ is difficult to obtain
- One notable exception: when $conv(S) = \overline{S}$; we then say that the MIP has the **integrality property**
- This condition is verified when all extreme points of $\overline{S}$ are integral
- Do you know a class of MIP models having the integrality property?

How to improve a MIP model

- Add constraints that:
  - Define valid inequalities for the feasible region (or for the set of optimal solutions)
  - Are not satisfied by some (fractional) solution to the LP relaxation, i.e. are **cuts**
  - Ideally, define facets of the convex hull of the feasible region
- Reformulate the problem in a different variable space
- Combine the two previous ideas

Exemple 1: Uncapacitated Facility Location Problem (UFLP)

- K: set of customers
- J: set of locations for potential facilities
- $d_k > 0$: demand of customer k
- $f_j \geq 0$: fixed cost for opening facility at location j
- $c_{jk} \geq 0$: unit cost of satisfying the demand of customer k from facility at location j
- **Problem description**: Determine the locations of the facilities to satisfy customers' demands at minimum cost

Model 1

- y_j: 1, if location j is chosen for a facility, 0, otherwise
- x_{jk}: 1, if facility at location j is used to satisfy the demand of customer k, 0, otherwise

$$\min \sum_{j \in J} \sum_{k \in K} d_k c_{jk} x_{jk} + \sum_{j \in J} f_j y_j$$

$$\sum_{j \in J} x_{jk} = 1, \quad k \in K$$

$$\sum_{k \in K} x_{jk} \leq |K| y_j, \quad j \in J$$

$$x_{jk} \in \{0, 1\}, \quad j \in J, k \in K$$

$$y_j \in \{0, 1\}, \quad j \in J$$

Discussion on Model 1

- Is the model valid? Can we have $\sum_{k \in K} x_{jk} = 0$ if $y_j = 1$?

Discussion on Model 1

- Is the model valid? Can we have $\sum_{k \in K} x_{jk} = 0$ if $y_j = 1$?
- Can we replace $x_{jk} \in \{0, 1\}, j \in J, k \in K$ by $x_{jk} \geq 0, j \in J, k \in K$?

Discussion on Model 1

- Is the model valid? Can we have $\sum_{k \in K} x_{jk} = 0$ if $y_j = 1$?
- Can we replace $x_{jk} \in \{0, 1\}, j \in J, k \in K$ by $x_{jk} \geq 0, j \in J, k \in K$?
- **Single-assignment property**: when the integrality of x_{jk} is relaxed, there is an optimal solution to this relaxed problem such that each customer's demand is satisfied from a single facility
- How can we solve the LP relaxation of Model 1?

Discussion on Model 1

- Is the model valid? Can we have $\sum_{k \in K} x_{jk} = 0$ if $y_j = 1$?
- Can we replace $x_{jk} \in \{0, 1\}, j \in J, k \in K$ by $x_{jk} \geq 0, j \in J, k \in K$?
- **Single-assignment property**: when the integrality of x_{jk} is relaxed, there is an optimal solution to this relaxed problem such that each customer's demand is satisfied from a single facility
- How can we solve the LP relaxation of Model 1?
- Since $f_j \geq 0$, we must have an optimal solution such that $y_j = \frac{1}{|K|} \sum_{k \in K} x_{jk}$
- Apply this result to project out of the model variables y_j. How do we solve the resulting LP?

Improving Model 1

- Based on your method for solving the LP relaxation, show that the lower bound can be very weak

Improving Model 1

- Based on your method for solving the LP relaxation, show that the lower bound can be very weak
- Suggest constraints that can be added to improve the LP relaxation

Improving Model 1

- Based on your method for solving the LP relaxation, show that the lower bound can be very weak
- Suggest constraints that can be added to improve the LP relaxation

$$x_{jk} \leq y_j, \quad j \in J, k \in K$$

- Are they valid inequalities? Cuts? Facets of the convex hull?

Improving Model 1

- Based on your method for solving the LP relaxation, show that the lower bound can be very weak
- Suggest constraints that can be added to improve the LP relaxation

$$x_{jk} \leq y_j, \quad j \in J, k \in K$$

- Are they valid inequalities? Cuts? Facets of the convex hull?
- When these inequalities are added, there is an optimal solution to the LP relaxation that satisfies $y_j = \max_{k \in K} x_{jk} \geq \frac{1}{|K|} \sum_{k \in K} x_{jk}$

Model 2

$$\min \sum_{j \in J} \sum_{k \in K} d_k c_{jk} x_{jk} + \sum_{j \in J} f_j y_j$$

$$\sum_{j \in J} x_{jk} = 1, \quad k \in K$$

$$x_{jk} \leq y_j, \quad j \in J, k \in K$$

$$x_{jk} \in \{0, 1\}, \quad j \in J, k \in K$$

$$y_j \in \{0, 1\}, \quad j \in J$$

- How to solve the LP relaxation?

Exemple 2: Two-Echelon Uncapacitated Facility Location Problem (TUFLP)

- K: set of customers
- J: set of locations for potential facilities
- I: set of locations for potential depots
- $d_k > 0$: demand of customer k
- $f_j \geq 0$: fixed cost for opening facility at location j
- $g_i \geq 0$: fixed cost for opening depot at location i
- $c_{jk} \geq 0$: unit cost of satisfying the demand of customer k from facility at location j
- $e_{ij} \geq 0$: unit cost for sending flow from depot location i to facility location j
- **Problem description**: Determine the locations of the depots and the locations of the facilities to satisfy customers' demands at minimum cost, given that all flows must depart at the depot locations (each depot location has unlimited supply)

Model 1

- y_j: 1, if location j is chosen for a facility, 0, otherwise
- z_i: 1, if location i is chosen for a depot, 0, otherwise
- x_{jk}: 1, if facility at location j is used to satisfy the demand of customer k, 0, otherwise
- w_{ij}: flow from depot location i to facility location j

$$\min \sum_{j\in J}\sum_{k\in K} d_k c_{jk} x_{jk} + \sum_{j\in J} f_j y_j + \sum_{i\in I}\sum_{j\in J} e_{ij} w_{ij} + \sum_{i\in I} g_i z_i$$

$$\sum_{j\in J} x_{jk} = 1, \quad k \in K$$

$$x_{jk} \leq y_j, \quad j \in J, k \in K$$

$$x_{jk} \in \{0,1\}, \quad j \in J, k \in K$$

$$y_j \in \{0,1\}, \quad j \in J$$

$$\sum_{i\in I} w_{ij} = \sum_{k\in K} d_k x_{jk}, \quad j \in J$$

$$w_{ij} \leq (\sum_{k\in K} d_k) z_i, \quad i \in I, j \in J$$

$$w_{ij} \geq 0, \quad i \in I, j \in J$$

$$z_i \in \{0,1\}, \quad i \in I$$

Discussion on Model 1

- Is the model valid?

Discussion on Model 1

- Is the model valid?
- The linking constraints between the flow variables and the binary variables are weak. How can we improve them?

Discussion on Model 1

- Is the model valid?
- The linking constraints between the flow variables and the binary variables are weak. How can we improve them?
- Introduce the variables v_{ijk}: *fraction* of the demand d_k of customer k satisfied from facility j *and* depot i
- Relationships to existing variables:

$$x_{jk} = \sum_{i \in I} v_{ijk}, \quad j \in J, k \in K$$

$$w_{ij} = \sum_{k \in K} d_k v_{ijk}, \quad i \in I, j \in J$$

- Project out variables x_{jk} and w_{ij} from Model 1. What do you obtain?

Model 2

$$\min \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} d_k(c_{jk}+e_{ij})v_{ijk} + \sum_{j\in J} f_j y_j + \sum_{i\in I} g_i z_i$$

$$\sum_{i\in I}\sum_{j\in J} v_{ijk} = 1, \quad k \in K$$

$$\sum_{i\in I} v_{ijk} \le y_j, \quad j \in J, k \in K$$

$$\sum_{k\in K} d_k v_{ijk} \le (\sum_{k\in K} d_k) z_i, \quad i \in I, j \in J$$

$$v_{ijk} \ge 0, \quad i \in I, j \in J, k \in K$$

$$z_i \in \{0,1\}, \quad i \in I$$

$$y_j \in \{0,1\}, \quad j \in J$$

Discussion on Model 2

- So far, nothing is gained by introducing the new variables!
- But, we can add the following valid inequalities:

$$v_{ijk} \leq z_i, \quad i \in I, j \in J, k \in K$$

- Now, can you eliminate the weak linking constraints?

Discussion on Model 2

- So far, nothing is gained by introducing the new variables!
- But, we can add the following valid inequalities:

$$v_{ijk} \leq z_i, \quad i \in I, j \in J, k \in K$$

- Now, can you eliminate the weak linking constraints?
- We also have the following *depot-customer single-assignment property*: for each pair depot-customer (i, k) such that d_k is satisfied from depot i, there is an optimal solution with a single facility j that satisfies d_k from depot i
- Exploit this property to derive valid inequalities that further improve the LP relaxation

Discussion on Model 2

- So far, nothing is gained by introducing the new variables!
- But, we can add the following valid inequalities:

$$v_{ijk} \leq z_i, \quad i \in I, j \in J, k \in K$$

- Now, can you eliminate the weak linking constraints?
- We also have the following *depot-customer single-assignment property*: for each pair depot-customer (i, k) such that d_k is satisfied from depot i, there is an optimal solution with a single facility j that satisfies d_k from depot i
- Exploit this property to derive valid inequalities that further improve the LP relaxation

$$\sum_{j \in J} v_{ijk} \leq z_i, \quad i \in I, k \in K$$

- Do you think these valid inequalities are better or worse than the previous ones (i.e., $v_{ijk} \leq z_i$, $i \in I, j \in J, k \in K$)?

Model 3

$$\min \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} d_k(c_{jk}+e_{ij})v_{ijk} + \sum_{j\in J} f_j y_j + \sum_{i\in I} g_i z_i$$

$$\sum_{i\in I}\sum_{j\in J} v_{ijk} = 1, \quad k \in K$$

$$\sum_{i\in I} v_{ijk} \le y_j, \quad j \in J, k \in K$$

$$\sum_{j\in J} v_{ijk} \le z_i, \quad i \in I, k \in K$$

$$v_{ijk} \in \{0,1\}, \quad i \in I, j \in J, k \in K$$

$$y_j \in \{0,1\}, \quad j \in J$$

$$z_i \in \{0,1\}, \quad i \in I$$

Discussion on Model 3

- The LP relaxation of Model 3 is better than that of Models 1 and 2, which were equivalent
- The reformulation of the problem in a different variable space allowed us to add valid inequalities that improve the LP relaxation
- Maybe we can do better by using the following *depot-facility single assignment property*: for each facility, there is a single depot assigned to it
- This property is true only because the cost on (i, j, k) is decomposable by (i, j) and (j, k)
- Use this property to define new variables and additional valid inequalities

Improving Model 3

- u_{ij}: 1, if facility location j is assigned depot location i
- Relationship to existing variables:

$$y_j = \sum_{i \in I} u_{ij}, \quad j \in J$$

- Valid inequalities:

$$v_{ijk} \leq u_{ij}, \quad i \in I, j \in J, k \in K$$

- Simplify Model 3 to obtain a model with a better LP relaxation

Model 4

$$\min \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} d_k (c_{jk} + e_{ij}) v_{ijk} + \sum_{i \in I} \sum_{j \in J} f_j u_{ij} + \sum_{i \in I} g_i z_i$$

$$\sum_{i \in I} \sum_{j \in J} v_{ijk} = 1, \quad k \in K$$

$$v_{ijk} \leq u_{ij}, \quad i \in I, j \in J, k \in K$$

$$\sum_{j \in J} v_{ijk} \leq z_i, \quad i \in I, k \in K$$

$$v_{ijk} \in \{0, 1\}, \quad i \in I, j \in J, k \in K$$

$$u_{ij} \in \{0, 1\}, \quad i \in I, j \in J$$

$$z_i \in \{0, 1\}, \quad i \in I$$