Characterization and Computation of Real-Radical Ideals using SDP

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Outline

- Introduction
- Algebraic Geometry
- Moment Methods
- Our Contribution
- Examples
- Outlook and Future Research

Introduction

Task Formulation:

Given:

Set $\mathcal{S}$ described by

- Polynomial equalities: $p_i(x_1, \ldots, x_n) = p_i(x) = 0$
- Inequality constraints: $f_i(x_1, \ldots, x_n) = f_i(x) \leq 0$

Task:

- Compute all solutions in $v \in \mathcal{S}$
- Compute (low degree) polynomial equalities vanishing on all points in $\mathcal{S}$

Introduction

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Algebraic Reformulation:

- Find all points in a semi-algebraic set $\mathcal{S}$
- Compute a (total-degree) Gröbner basis of the ($\mathcal{S}$-radical) ideal $I(\mathcal{S})$

Introduction

Example: Numerical Integration (Gauss Quadrature Formula) with two weights w_1, w_2 and two knots y_1, y_2.

$$\boxed{\int_{-1}^{1} f(z)dz = w_1 f(y_1) + w_2 f(y_2) \quad \forall f(z) \in \{\text{poly. of degree} \leq 3\}}$$

$x = [w_1, w_2, y_1, y_2]$ has to satisfy:

$$p_1 = x_1 + x_2 - 2 = 0\,,$$
$$p_2 = x_1 x_3 + x_2 x_4 = 0\,,$$
$$p_3 = x_1 x_3^2 + x_2 x_4^2 - \frac{2}{3} = 0\,,$$
$$p_4 = x_1 x_3^3 + x_2 x_4^3 = 0$$

and

$$-1 \leq x_i \leq 1 \; \forall\, i\,.$$

- Relatively "high" degree (d=4)
- Equalities and inequalities

=> find solutions

=> find low degree polynomial equalities "describing" the same set

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$$p_3 = x_1 x_3^2 + x_2 x_4^2 - \frac{2}{3} = 0\,,$$
$$p_4 = x_1 x_3^3 + x_2 x_4^3 = 0$$

and

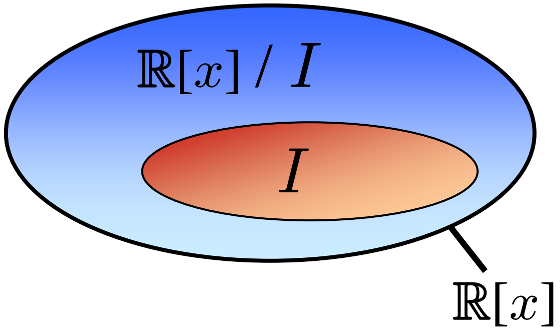
$$-1 \leq x_i \leq 1 \;\forall\, i\,.$$

Corresponding $\mathcal{S}$-radical ideal:

$$g_1 = x_1 - 1 = 0\,,$$
$$g_2 = x_2 - 1 = 0\,,$$
$$g_3 = x_3^2 - \frac{1}{3} = 0\,,$$
$$g_4 = x_3 + x_4 = 0\,,$$

$$V_{\mathcal{S}}(I) = \{(1, 1, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (1, 1, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\}$$

Algebraic Geometry - Preliminaries



- Ring of Polynomials: *"All polynomials"*

  $$\mathbb{R}[x_1, \ldots, x_n] = \mathbb{R}[x]$$

- Ideal: *"A set of polynomials { $f(x_1, \ldots, x_n)$ }"*

  $$I = \langle p_1, \ldots, p_m \rangle = \left\{ f \in \mathbb{R}[x_1, \ldots, x_n] \;\middle|\; f = \sum_{i=1}^{m} p_i \phi_i \,,\, \phi_i \in \mathbb{R}[x_1, \ldots, x_n] \right\}$$

  with $\phi_j(x) \in \mathbb{R}[x]$ arbitrary

- Quotient Ring: *"Set of all potential remainders"*

  $$\mathbb{R}[x]/I \;:\; [a] = a + I := \{a + r \;:\; r \in I\}$$

Algebraic Geometry - Preliminaries

Given:

A Set of Polynomials: $p_i \in \mathbb{R}[x_1, \ldots, x_n]\,, \quad p_i = \text{vec}(p_i)^T \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x^d \end{pmatrix}$

$x \in \mathbb{R}^n$

Multi-index notation:

$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$

Algebraic Geometry:

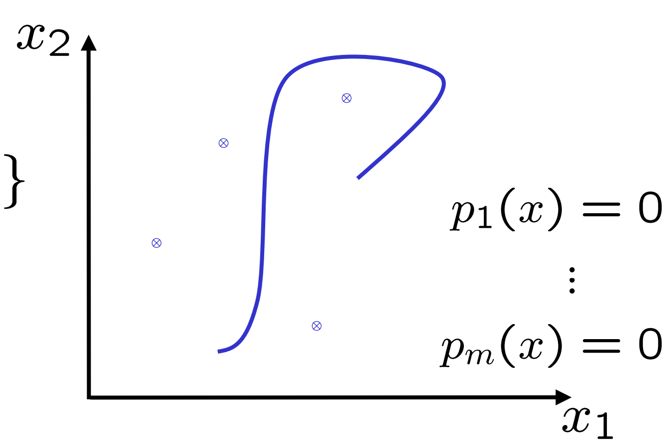
- Ideal:

$$I = \langle p_1, \ldots, p_m \rangle = \left\{ f \in \mathbb{R}[x_1, \ldots, x_n] \,\middle|\, f = \sum_{i=1}^{m} p_i \phi_i \,, \phi_i \in \mathbb{R}[x_1, \ldots, x_n] \right\}$$

$$\Downarrow$$

- Variety (“The Zero-Set”):

$$V(I) = \{ v \in \mathbb{C}^n \,|\, p_i(v) = 0, i = 1, \ldots, m \}$$



Zero-dimensional Varieties

Algebraic Geometry - Preliminaries

Given:

A Set of Polynomials: $p_i \in \mathbb{R}[x_1, \ldots, x_n]\,, \quad p_i = \text{vec}(p_i)^T \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x^d \end{pmatrix}$

$x \in \mathbb{R}^n$

Multi-index notation:

$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$

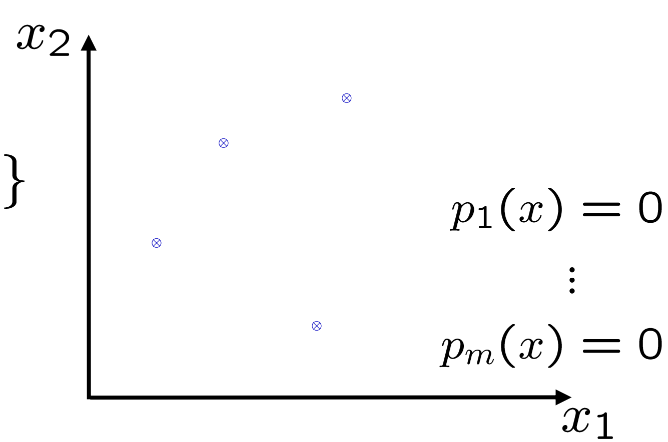
Algebraic Geometry:

- Radical Ideal:

$$I(V) = \sqrt{I} = \Big\{ \quad \text{all polynomials vanishing precisely on } v \in V(I) \quad \Big\}$$

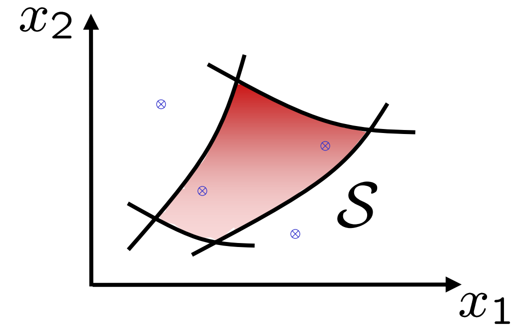
- Variety (“The Zero-Set”):

$$V(I) = \{v \in \mathbb{C}^n \,|\, p_i(v) = 0, i = 1, \ldots, m\}$$



Algebraic Geometry - The real case

Our aim: Find generators $g_1, g_2, \ldots$ (i.e. defining equations) for the vanishing ideal



- of all solutions, $V(I) \subset \mathbb{C}^n$

  ⇨ Radical Ideal: $I(V(I)) = \langle g_1, g_2, \ldots, \rangle$

- of real solutions, $V_{\mathbb{R}}(I) = V(I) \cap \mathbb{R}^n$

  ⇨ Real Radical: $I(V_{\mathbb{R}}(I)) = \langle g_1, g_2, \ldots, \rangle$

- of solutions on $\mathcal{S}$, $V_{\mathcal{S}}(I) = V_{\mathbb{R}}(I) \cap \mathcal{S}$

  ⇨ S-Radical: $I(V_{\mathcal{S}}(I)) = \langle g_1, g_2, \ldots, \rangle$

Introduction
- Existing Methods

Complex Case $I(V(I))$ well understood:

- Mostly based on extensive Gröbner Basis Computation
- Numerical methods (e.g. Homotopy [Sommese, Wampler])

Real Algebraic Geometry Methods

- e.g. quantifier elimination, CAD
  => high complexity

General trend in applied polynomial algebra:

$\Rightarrow$ Numerical Polynomial Algebra

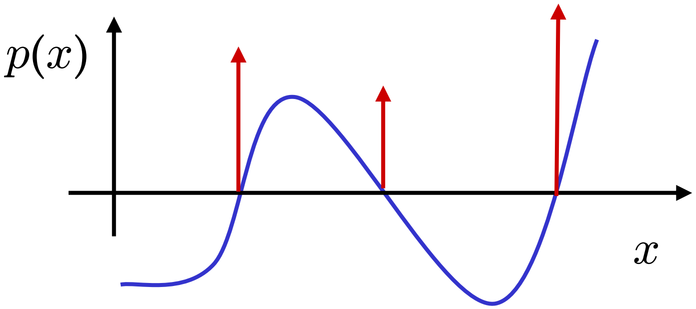
i) Algebraic methods ([Reid,Zhi], [S.,W.],…)

ii) Real Algebraic methods (SOS/Moment)

Moment methods -Motivation

Lets start with:

$$V_{\mathbb{R}}(I) = \{v \in \mathbb{R}^n \,|\, p_i(v) = 0, i = 1, \ldots, m\}$$



Characterization via finite atomic probability measures:

$$K^{\mu} = \{\mu \mid \mu \in \mathcal{P}, \mu(V_{\mathbb{R}}(I)) = 1, \mu(\mathbb{R}^n / V_{\mathbb{R}}(I)) = 0\}$$

Moment methods -Motivation

Lets start with:

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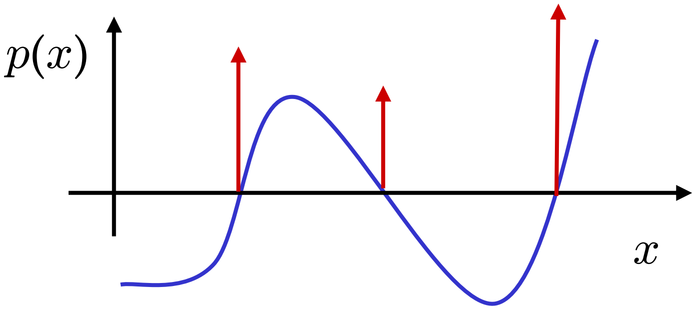


Characterization via finite atomic probability measures:

$$K^{\mu} = \{\mu \mid \mu \in \mathcal{P}, \int_{\mu} p_i(x)\mu(dx) = 0$$
$$\int_{\mu} x_j p_i(x)\mu(dx) = 0, \ldots,$$
$$\int_{\mu} x^{\beta} p_i(x)\mu(dx) = 0, \ldots \}$$

Moment methods
-Sequence of Moments

- Moment of order α

$$y_\alpha = \int_\mu x^\alpha \mu(dx)$$

with μ being some positive Borel measure

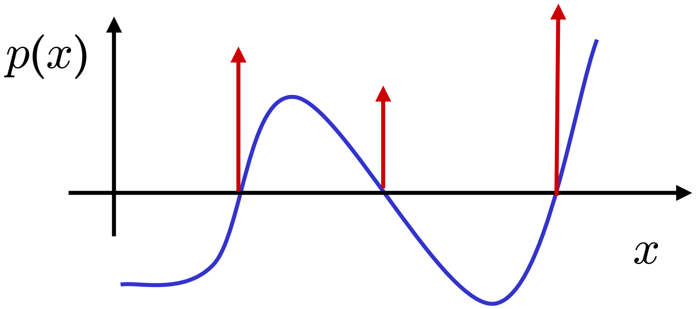
- Sequence of Moments

$$\{y_\alpha^\mu\}_{\alpha=0}^\infty \in Y_\mathcal{P} \subseteq \mathbb{R}^{N^n}$$

Moment methods -Motivation

Lets start with:

$$V_{\mathbb{R}}(I) = \{v \in \mathbb{R}^n \,|\, p_i(v) = 0, i = 1, \ldots, m\}$$



Characterization via finite atomic probability measures:

$$K^y = \Big\{y \mid y \in Y_{\mathcal{P}}, \sum_\alpha p_\alpha y_\alpha = 0,$$
$$\sum_\alpha p_\alpha y_{\alpha+e_i} = 0, \ldots,$$
$$\sum_\alpha p_\alpha y_{\alpha+\beta} = 0, \ldots \Big\}$$

Moment methods
-The moment matrix

- Moment of order α

$$y_\alpha = \int_\mu x^\alpha \mu(dx)$$

with μ being some positive Borel measure

- Sequence of Moments

$$\{y_\alpha^\mu\}_{\alpha=0}^\infty \in Y_\mathcal{P} \subseteq \mathbb{R}^{N^n}$$

- Moment matrix

$$M_d(y) = \begin{array}{cc} \begin{array}{ccccc} 1 & x_1 & x_2 & x_1^2 & \cdots \end{array} & \\ \begin{pmatrix} 1 & y_{10} & y_{01} & \cdots & y_d \\ y_{10} & y_{20} & y_{11} & & \\ y_{01} & y_{11} & y_{02} & & \\ \vdots & & & \ddots & \\ y_d & & & & y_{d+d} \end{pmatrix} & \begin{array}{c} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ \vdots \end{array} \end{array}$$

Moment methods
-The moment matrix

- Moment of order α

$$y_\alpha = \int_\mu x^\alpha \mu(dx)$$

with μ being some positive Borel measure

- Sequence of Moments

$$\{y_\alpha^\mu\}_{\alpha=0}^\infty \in Y_\mathcal{P} \subseteq \mathbb{R}^{N^n}$$

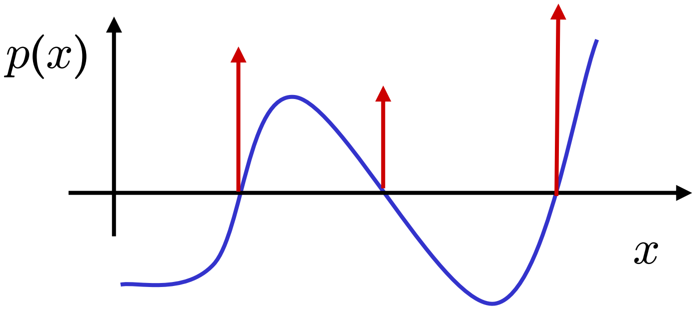
- Moment matrix

$$M_d(y) = \begin{pmatrix} 1 & y_{10} & y_{01} & \cdots & y_d \\ y_{10} & y_{20} & y_{11} & & \\ y_{01} & y_{11} & y_{02} & & \\ \vdots & & & \ddots & \\ y_d & & & & y_{d+d} \end{pmatrix} \succeq 0 \text{ if } \mu \in \mathcal{P}$$

Moment methods
-Motivation

Lets start with:

$$V_{\mathbb{R}}(I) = \{v \in \mathbb{R}^n \,|\, p_i(v) = 0, i = 1, \ldots, m\}$$



Characterization via finite atomic probability measures:

$$K^y = \Big\{y \mid y \in Y_{\mathcal{P}}, \sum_\alpha p_\alpha y_\alpha = 0,$$

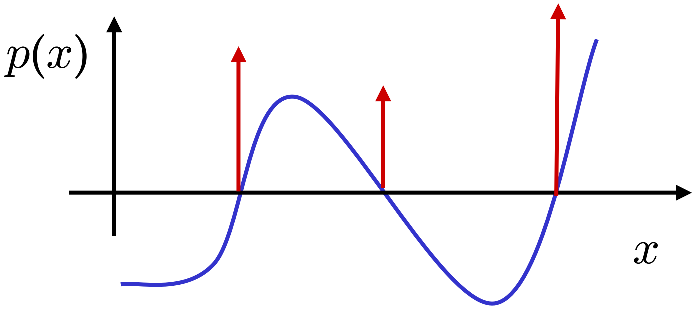
$$\sum_\alpha p_\alpha y_{\alpha+e_i} = 0, \ldots,$$

$$\sum_\alpha p_\alpha y_{\alpha+\beta} = 0, \ldots\Big\}$$

Moment methods - Relaxation

Lets start with:

$$V_{\mathbb{R}}(I) = \{v \in \mathbb{R}^n \,|\, p_i(v) = 0, i = 1, \ldots, m\}$$



Relaxation

~~Characterization~~ via finite atomic probability measures:

$$K = \{y \,|\, y \in \mathbb{R}^{N_\infty}, M(y) \succeq 0, \sum_\alpha p_\alpha y_\alpha = 0,$$
$$\sum_\alpha p_\alpha y_{\alpha+e_i} = 0, \ldots,$$
$$\sum_\alpha p_\alpha y_{\alpha+\beta} = 0, \ldots\}$$

but, if dimension of $V_{\mathbb{R}}(I)$ is zero, then this relaxation is tight.

Moment methods - Truncation

Use truncated moment matrices $M_t(y)$ and constraints:

$$K_t = \{y \mid y \in \mathbb{R}^{N_t}, M_t(y) \succeq 0, \sum_\alpha p_\alpha y_\alpha = 0,$$

Only constraints, that can be expressed with moments of order $\leq$ t

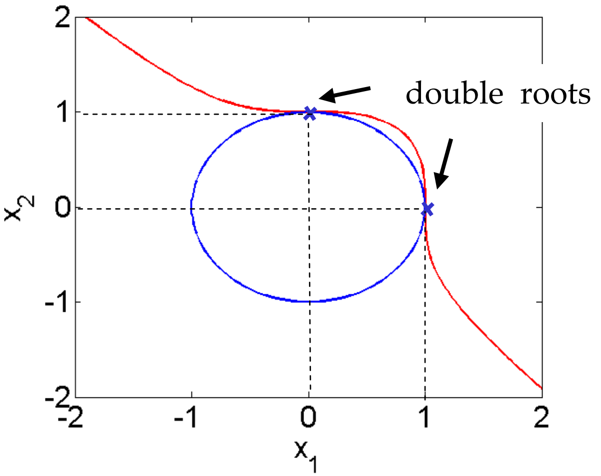
$$\sum_\alpha p_\alpha y_{\alpha+e_i} = 0, \ldots, \sum_\alpha p_\alpha y_{\alpha+\beta} = 0\}$$

Thm. [based on Curto and Fialkow]:

(F1) rank $(M_s(y))$ = rank $(M_{s-d}(y))$ for some $d \leq s \leq t$
(F2) rank $(M_s(y))$ = rank $(M_{s-1}(y))$ for some $2d \leq s \leq t$
with $d = \max_{\text{all } p} (\lceil \deg p / 2 \rceil)$

If (F1) or (F2) holds, then $M_s(y)$ can be used to compute $W \subseteq V_{\mathbb{R}}(I)$, where rank $(M_s(y)) = \#W$.

Moment method
- Running Examples



$$p_1(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$$
$$p_2(x_1, x_2) = x_1^3 + x_2^3 - 1 = 0$$

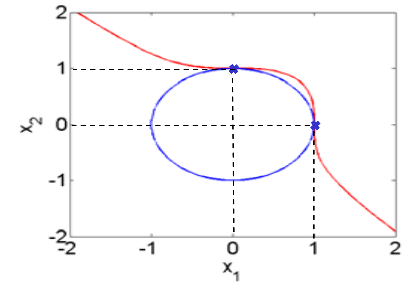
- Variety $V(I)$ of $I = <p_1, p_2>$

$$V(I) = \Big\{(-1 + i\frac{\sqrt{2}}{2}, -1 - i\frac{\sqrt{2}}{2}), (-1 - i\frac{\sqrt{2}}{2}, -1 + i\frac{\sqrt{2}}{2}), (1,0), (0,1)\Big\}$$

- Real Variety $V_{\mathbb{R}}(I)$ of $I = <p_1, p_2>$

$$V_{\mathbb{R}}(I) = \Big\{(1,0), (0,1)\Big\}$$

Moment method
- Running Examples



$$p_1(x_1,x_2) = x_1^2 + x_2^2 - 1 = 0$$
$$p_2(x_1,x_2) = x_1^3 + x_2^3 - 1 = 0$$

$K_2 = \{\ (1, y_{10}, y_{01}, \ldots)\ |$

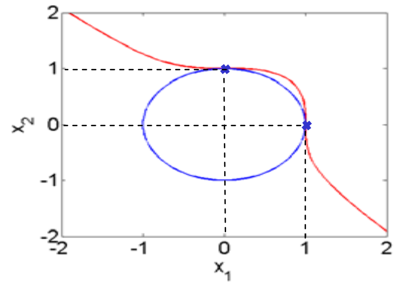
$$M_0(y) \qquad M_1(y)$$

$$M_2(y) = \begin{pmatrix} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{11} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{pmatrix} \succeq 0,$$

$$\begin{aligned} y_{20} + y_{02} - 1 &= 0, \\ y_{30} + y_{12} - y_{10} &= 0, \\ y_{21} + y_{03} - y_{01} &= 0, \\ y_{40} + y_{22} - y_{20} &= 0, \\ y_{31} + y_{13} - y_{11} &= 0, \\ y_{22} + y_{04} - y_{02} &= 0, \end{aligned} \qquad \begin{aligned} y_{30} + y_{03} - 1 &= 0, \\ y_{40} + y_{13} - y_{10} &= 0, \\ y_{31} + y_{04} - y_{01} &= 0\} \end{aligned}$$

$$\int_\mu x_1 p_2(x)\mu(dx) = 0$$

Moment method
- Running Examples



$$p_1(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$$
$$p_2(x_1, x_2) = x_1^3 + x_2^3 - 1 = 0$$

$$M_2(y) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \underset{1,\, x_1,\, x_2,\, x_1^2,\, \ldots}{\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}}$$

Use (F1):

$d = \max(\lceil \deg p / 2 \rceil)$

rank $(M_2(y))$ = rank $(M_0(y))$ = 1 for some $2 \leq s \leq t$

s $\quad$ s-d

$W = \{(x_1, x_2) = (0,1)\}$

Moment method - Open issues

- How can we make sure that we found all solutions?
- How do we get generators for $I(V_{\mathbb{R}}(I))$?

Answer: Use insight from Algebraic Geometry

Thm.:

$$< \mathbb{R}^{N_s} / \ker M_s > \;\hat{=}\; \mathbb{R}[x] / I(V_{\mathbb{R}})$$

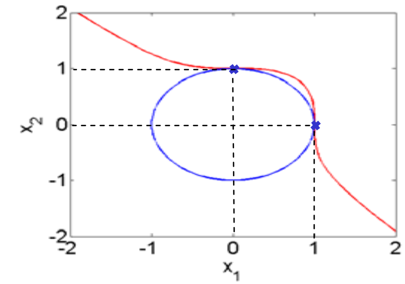
$$<\ker M_s> \qquad I(V_{\mathbb{R}})$$

$\mathbb{R}[x]$ $\qquad$ $\mathbb{R}[x]$

iff (F1) or (F2)
and rank $M_s(y)$ is maximum
among all $y \in K_s$

Here <ker M_s> denotes all polynomials p(x) with vec(p) $\in$ ker M_s (resp. vec(p) $\in$ range M_s)

Our contribution
- Running Examples



$$p_1(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$$
$$p_2(x_1, x_2) = x_1^3 + x_2^3 - 1 = 0$$

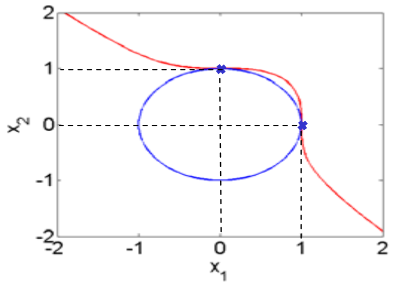
$K_2 = \{ (1, y_{10}, y_{01}, \ldots) \mid$

$$\begin{array}{c} \qquad\qquad M_0(y) \qquad\qquad M_1(y) \\ M_2(y) = \begin{pmatrix} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{11} & y_{21} & & & \\ & & & & y_{22} & y_{13} \\ & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{pmatrix} 0, \end{array}$$

But now with: rank $M_2(y)$ is maximum for all $y \in K_2$

$$y_{20} + y_{02} - 1 = 0,$$
$$y_{30} + y_{12} - y_{10} = 0,$$
$$y_{21} + y_{03} - y_{01} = 0,$$
$$y_{40} + y_{22} - y_{20} = 0,$$
$$y_{31} + y_{13} - y_{11} = 0,$$
$$y_{22} + y_{04} - y_{02} = 0,$$
$$y_{30} + y_{03} - 1 = 0,$$
$$y_{40} + y_{13} - y_{10} = 0,$$
$$y_{31} + y_{04} - y_{01} = 0\}$$

Our contribution
- Running Examples



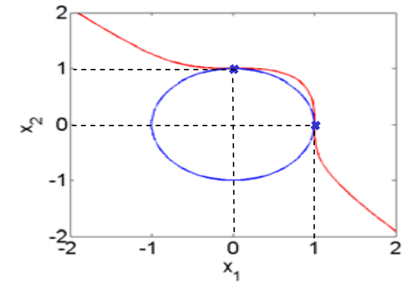
$$p_1(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$$
$$p_2(x_1, x_2) = x_1^3 + x_2^3 - 1 = 0$$

$$M_2(y) = \begin{pmatrix} 1 & 0.5 & 0.5 & 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

Now rank $(M_2(y)) = 2$ i.e. we need rank$(M_3(y)) = 2$, (which happens to be the case)

$W = \{(1,0), (0,1)\} = V_{\mathbb{R}}(I)$

Our contribution
- Running Examples



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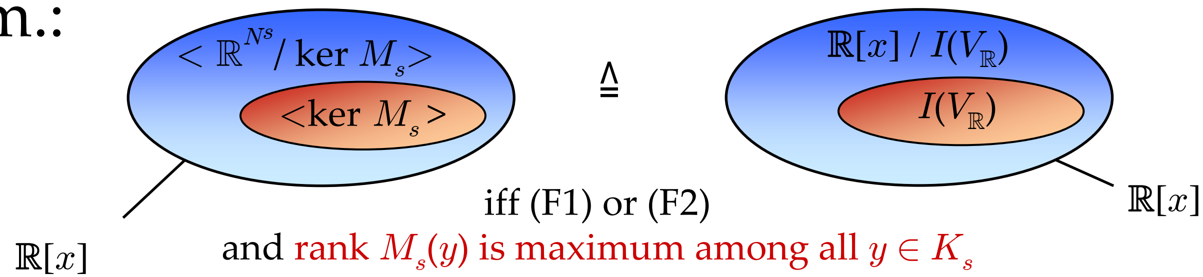
Thm.:

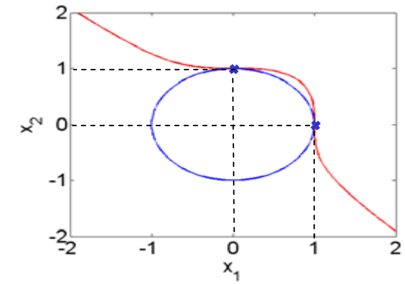


iff (F1) or (F2)

and rank $M_s(y)$ is maximum among all $y \in K_s$

$$0 = \begin{array}{c} \begin{pmatrix} 1 & 0.5 & 0.5 & 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \end{pmatrix} \\ \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{array} \end{array} \mathrm{vec}(g)$$

Our contribution - Running Examples



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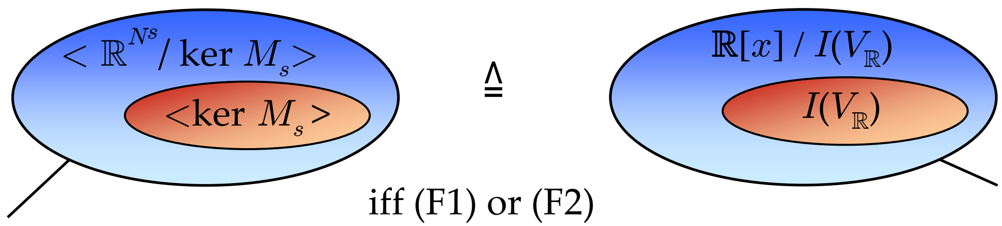
$$p_2(x_1, x_2) = x_1^3 + x_2^3 - 1 = 0$$

Thm.:



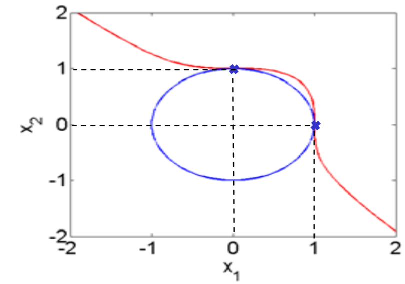
$\mathbb{R}[x]$ $\mathbb{R}[x]$

iff (F1) or (F2)

and rank $M_s(y)$ is maximum among all $y \in K_s$

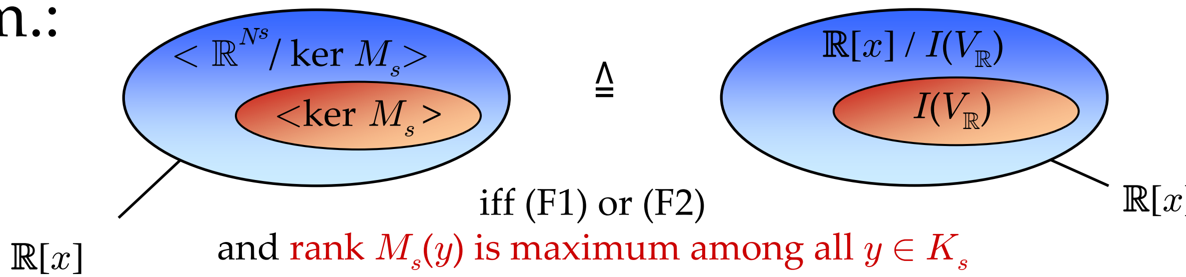
$$0 = \begin{array}{c} \begin{pmatrix} 1 & 0.5 & 0.5 & 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \end{pmatrix} \\ \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{array} \end{array} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} g_1(x_1, x_2) = x_2 + x_1 - 1 \\ \in I(V_\mathbb{R}(I)) \end{array}$$

Our contribution
- Running Examples



$$p_1(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$$
$$p_2(x_1, x_2) = x_1^3 + x_2^3 - 1 = 0$$

Thm.:



iff (F1) or (F2)

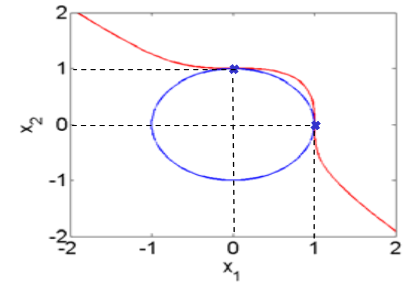
and rank $M_s(y)$ is maximum among all $y \in K_s$

$$0 = \begin{array}{cccccc} \left(\begin{array}{cccccc} 1 & 0.5 & 0.5 & 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \end{array}\right) \\ \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{array} \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad g_1(x_1, x_2) = x_2 + x_1 - 1 \in I(V_{\mathbb{R}}(I))$$

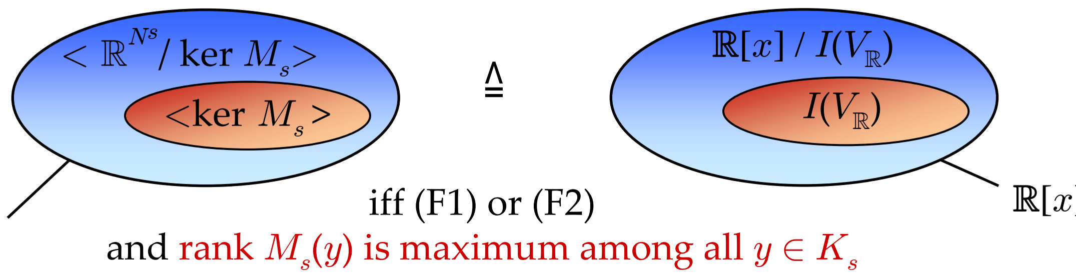
$$\Rightarrow \quad g_2(x_1, x_2) = x_1 x_2 \in I(V_{\mathbb{R}}(I))$$

Our contribution - Running Examples



$$p_1(x_1,x_2) = x_1^2 + x_2^2 - 1 = 0$$
$$p_2(x_1,x_2) = x_1^3 + x_2^3 - 1 = 0$$

Thm.:



iff (F1) or (F2)

and rank $M_s(y)$ is maximum among all $y \in K_s$

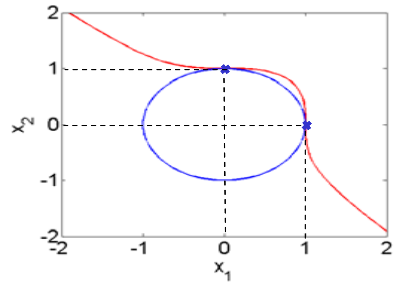
$$0 = \begin{array}{cccccc} \multicolumn{6}{c}{\left(\begin{array}{cccccc} 1 & 0.5 & 0.5 & 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \end{array}\right)} \\ 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{array} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $g_1(x_1,x_2) = x_2 + x_1 - 1 \in I(V_{\mathbb{R}}(I))$

$\Rightarrow$ $g_2(x_1,x_2) = x_1x_2 \in I(V_{\mathbb{R}}(I))$

$\Rightarrow$ $g_3(x_1,x_2) = x_1^2 - x_1 \in I(V_{\mathbb{R}}(I))$

Our contribution
- Running Examples



$$p_1(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$$
$$p_2(x_1, x_2) = x_1^3 + x_2^3 - 1 = 0$$

Some polynomials in the ideal:

$$g_1(x_1, x_2) = x_2 + x_1 - 1 \in I(V_{\mathbb{R}}(I))$$
$$g_2(x_1, x_2) = x_1 x_2 \in I(V_{\mathbb{R}}(I))$$
$$g_3(x_1, x_2) = x_1^2 - x_1 \in I(V_{\mathbb{R}}(I))$$

$$\Rightarrow$$

Evaluated at $V_{\mathbb{R}}$:

$$\forall v \in V_{\mathbb{R}}(I):$$
$$g_1(v_1, v_2) = v_2 + v_1 - 1 = 0$$
$$g_2(v_1, v_2) = v_1 v_2 = 0$$
$$g_3(v_1, v_2) = v_1^2 - v_1 = 0$$

Some rewriting:

$$\forall v \in V_{\mathbb{R}}(I):$$
$$v_2 = -v_1 + 1$$
$$v_1 v_2 = 0$$

Eigenvalue Problem:

$$\forall v \in V_{\mathbb{R}}(I):$$
$$v_2 \begin{pmatrix} 1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ v_1 \end{pmatrix}$$

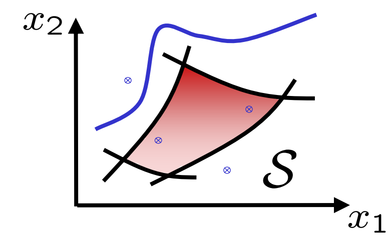
$\mathcal{X}_2$: Generalized Companion matrices

$$\lambda, \begin{pmatrix} 1 \\ v_1 \end{pmatrix} = \left\{ 0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}; 1, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

Our contribution - Summary

- A semi-definite characterization of $I(V_{\mathbb{R}}(I))$ [as the kernel of a moment matrix]
- An Algorithm for finding a basis (border/ Gröbner for total degree term ordering) of $I(V_{\mathbb{R}}(I))$, a linear basis of $\mathbb{R}[x]/I(V_{\mathbb{R}}(I))$ as well as the desired set $V_{\mathbb{R}}(I)$

Remarks about the method:



- real-algebraic in nature no complex roots computed
- finite convergence
- numerical based on SDP and numerical linear algebra
- extends to $I(V_{\mathcal{S}}(I))$=$I(V_{\mathbb{R}}(I) \cap \mathcal{S})$, where $\mathcal{S} = \{x \mid h_j(x) \geq 0 \}$
- works if $V(I)$ is infinite as long as $V_{\mathbb{R}}(I)$ resp. $I(V_{\mathcal{S}}(I))$ is finite

Our contribution
- Some algorithmic issues

How to find $y \in K_t$ for which rank $M_t(y)$ is maximum?

We use a SDP solver with 'extended self dual embedding' [e.g. SeDuMi-1.05]

How to compute the rank of a possibly ill conditioned matrix?

We use SVD, but this is the most sensitive numerical issue…

Example
- Katsura 5 (PoSSo test suite)

A problem of magnetism

[S. Katsura, 1990] "Spin Glass Problem by the Method of Integral Equation of the Effective Field". In "New Trends in Magnetism", edited by Mauricio D. Coutinho-Filho and Sergio M. Resende, pages 110-121, World Scientific, 1990.

$$
\begin{aligned}
p_1 &= 2x_6^2 + 2x_5^2 + 2x_4^2 + 2x_3^2 + 2x_2^2 + x_1^2 - x_1 = 0 \\
p_2 &= x_6x_5 + x_5x_4 + 2x_4x_3 + 2x_3x_2 + 2x_2x_1 - x_2 = 0 \\
p_3 &= 2x_6x_4 + 2x_5x_3 + 2x_4x_2 + x_2^2 + 2x_3x_1 - x_3 = 0 \\
p_4 &= 2x_6x_3 + 2x_5x_2 + 2x_3x_2 + 2x_4x_1 - x_4 = 0 \\
p_5 &= x_3^2 + 2x_6x_1 + 2x_5x_1 + 2x_4x_1 - x_5 = 0 \\
p_6 &= 2x_6 + 2x_5 + 2x_4 + 2x_3 + 2x_2 + x_1 - 1 = 0
\end{aligned}
$$

$$
\begin{aligned}
V_{\mathbb{R}}(I) = \{&(0.277, 0.226, 0.162, 0.0858, 0.0115, -0.124), \\
&(0.59, 0.0422, 0.327, -0.0642, -0.0874, -0.0132), \\
&(1, -2.8\text{e-}7, 4.7\text{e-}7, 8.81\text{e-}7, -2.79\text{e-}6, -3.69\text{e-}6), \\
&(0.239, 0.0608, -0.0622, -0.0233, 0.186, 0.219), \\
&(0.441, 0.151, 0.0225, 0.219, 0.0935, -0.207), \\
&(0.726, -0.0503, 0.122, 0.164, 0.11, -0.208), \\
&(0.462, 0.309, 0.0553, -0.102, -0.0844, 0.0917), \\
&(0.292, -0.101, 0.181, -0.0591, 0.193, 0.141), \\
&(0.753, 0.0532, 0.191, -0.114, -0.146, 0.139), \\
&(0.409, -0.0732, 0.0657, -0.127, 0.252, 0.178), \\
&(0.68, 0.266, -0.154, 0.0323, 0.0897, -0.0735), \\
&(0.136, 0.0428, 0.0417, 0.0404, 0.0964, 0.211)\}
\end{aligned}
$$

order t	rank sequence	extract. order MON/SVD	accuracy MON/SVD	comm. error MON/SVD
1	1 7	—	—	—
2	1 6 16	—	—	—
3	1 6 12 12	—/3(3)	—/1.1928e-005	—/2.3073e-007

Example
- Bini-Mourrain test suite

Numerical Bisection

[R.B. Kearfott, 1987] "Some tests of generalized bisection. ACM Transactions on Mathematical Software, 13(3):197--220, 1987.

Number of solutions: $|V(I)| = 20$
Number of real solutions: $|V_{\mathbb{R}}(I)| = 8$

$$h_1 = 5x_1^9 - 6x_1^5 x_2 + x_1 x_2^4 + 2x_1 x_3 = 0$$
$$h_2 = -2x_1^6 x_2 + 2x_1^2 x_2^3 + 2x_2 x_3 = 0$$
$$h_3 = x_1^2 + x_2^2 - 0.265625 = 0$$

$$\begin{aligned} V_{\mathbb{R}}(I) = \{&(-0.515, -0.000153, -0.0124), \\ &(-0.502, 0.119, 0.0124), \\ &(0.502, 0.119, 0.0124), \\ &(0.515, -0.000185, -0.0125), \\ &(0.262, 0.444, -0.0132), \\ &(-2.07\text{e-}5, 0.515, -1.27\text{e-}6), \\ &(-0.262, 0.444, -0.0132), \\ &(-1.05\text{e-}5, -0.515, -7.56\text{e-}7)\} \end{aligned}$$

order t	rank sequence	extract. order MON/SVD	accuracy MON/SVD	comm. error MON/SVD
5	1 4 8 16 25 34	—	—	—
6	1 3 9 15 22 26 32	—	—	—
7	1 3 8 10 12 16 20 24	3(3)/—(—)	0.12786/—	0.00019754/—
8	1 4 8 8 8 12 16 20 24	4(3)/3(3)	4.6789e-5/0.00013406	4.7073e-5/0.00075005

Outlook
- Current and Future Research

Ongoing Research:

- Positive dimensional ideals ($|V_{\mathbb{R}}(I)| = \infty$)
- Finding all complex solutions using numerical linear algebra only (no SDP)
- Error estimates

  e.g. zero cluster, condition number
- Improved accuracy

  e.g. use of scaling, SVD from SeDuMi

Done! Questions?

That was

"Characterization and computation of real radical ideals using SDP!"