Cut-First Branch-and-Price-Second for the CARP

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Capacitated Arc-Routing Problem (CARP)

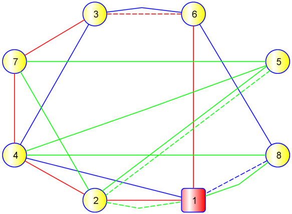
   

Notation: Defined on undirected Graph $G = (V, E)$

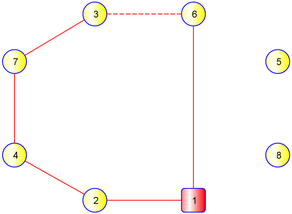
- demands $q_e \geq 0$ on edges $e \in E$
- required edges $E_R = \{e \in E : q_e > 0\} \subseteq E$
- K homogeneous vehicles stationed at depot d with capacity Q
- c_e cost for traversing through $e \in E$

Task: Find K minimum cost vehicles tours which start and end at depot d, service all required edges E_R, and respect the vehicle capacity Q.

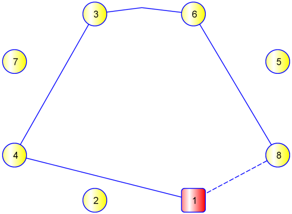
CARP solutions



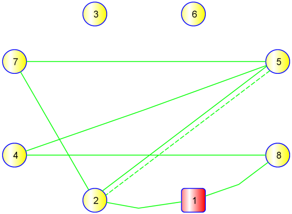
Complete solution



Route 1



Route 2



Route 3

Letchford, A. N. and Oukil, A. (2009). Exploiting sparsity in pricing routines for the capacitated arc routing problem, *Computers & Operations Research* **36**(7), 2320–2327.

- An approach that fully **exploits sparsity** of CARP instances should be superior
- Drawbacks of the proposed CG algorithm:
  1. Pure set-partitioning master program delivers worse bounds than known compact formulations
  2. Missing branching scheme

Branch-and-price for CARP is different than the node-routing case!

Two types of exact approaches:

1. Full exact methods determine an optimal integer solution and prove optimality by showing that its cost is a lower bound.
2. LB-based methods use heuristic solutions (often not computed within the presented approach) to prove optimality by showing that a computed lower bound matches with the heuristic's upper bound.

Methods presented in the Literature:

- **Two-Index Formulation** (Belenguer and Benavent, 1998)
- **One-Index Formulation** (Letchford, 1997; Belenguer and Benavent, 1998, 2003)
- **Set-Partitioning Formulation** (Gómez-Cabrero *et al.*, 2005)
- Transformation into Node-Routing Problem

Cut-First Branch-and-Price-Second

Key ideas:

- use strong LB from one-index formulation
- use aggregated set-partitioning formulation to avoid symmetric solutions
- exploit sparsity of real-world CARP graph;
  fast pricing on sparse network

Difficulties that had to be solved:

- negative reduced costs on deadheading edges (→dual optimal inequalities; Ben Amor *et al.* (2006))
- integer variables of aggregated formulation alone do not ensure integrality of tours
  (→complex branching scheme)

Overview: Cut-first bap-second

Input: CARP Instance

1. Phase 1: Cut
   Solve LP-relaxation of one-index formulation using a cutting-plane algorithm
   Let $\mathcal{S}$ be the set of active cuts at the end
   (odd-set, capacity, and disjoint-path inequalities)
2. Phase 2: Branch-and-Price
   Solve integer master program using CG und B&B
   Initialize MP with cuts $\mathcal{S}$

Output: Integer solution

Master problem derived from Dantzig-Wolfe decomposition of two-index formulation + cuts from one-index formulation + aggregation over vehicles:

$$
\begin{aligned}
\min \quad & \sum_{r \in \Omega} c_r^\top \lambda_r \\
\text{s.t.} \quad & \sum_{r \in \Omega} \bar{x}_{er} \lambda_r = 1 \quad \text{for all } e \in E_R \qquad (\text{dual } \pi_e) \\
& \sum_{r \in \Omega} d_{sr} \lambda_r \geq r_s \quad \text{for all } s \in \mathcal{S} \qquad (\text{dual } \beta_s) \\
& \sum_{r \in \Omega} \lambda_r = K \qquad (\text{dual } \mu) \\
& \lambda \geq \mathbf{0} \quad (\in \mathbb{R}^{|\Omega|})
\end{aligned}
$$

Reduced Costs (rdc) can be transformed back onto variables of the compact formulation:

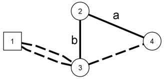
$$\tilde{c}_e^{serv} = c_e^{serv} - \pi_e \text{ for all } e \in E_R \qquad \text{and} \qquad \tilde{c}_e = c_e - \sum_{s \in \mathcal{S}} d_{es}\beta_s \text{ for all } e \in E.$$

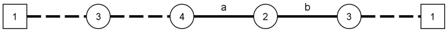
Dual optimality inequalities guarantee non-negative rdc $\tilde{c}_e \geq 0$ for deadheading (Bode and Irnich, 2012).

Pricing Problem:

$$\begin{aligned} \min \quad & \tilde{c}^{serv,\top} x + \tilde{c}^\top y - \mu \\ \text{s.t.} \quad & x(\delta_R(i)) + y(\delta(i)) = 2p_i \quad \text{for all } i \in V \\ & q^\top x \leq Q \\ & x(\delta_R(S)) + y(\delta(S)) \geq 2x_f \quad \text{for all } S \subseteq V \setminus \{d\},\ f \in E_R(S) \\ & p \in \mathbb{Z}_+^{|V|}, y \in \mathbb{Z}_+^{|E|}, x \in \{0,1\}^{|E_R|}, \end{aligned}$$

Example: $x_{23} = x_{24} = 1, y_{13} = 2, y_{34} = 1$ implies tour $1 - 3 - 4 \overset{a}{=} 2 \overset{b}{=} 3 - 1$





Pricing can be done on the **original network** (exploit sparsity, Letchford and Oukil (2009))

Extension along deadheading edges can be done with the Dijkstra Algorithm due to non-negative reduced costs $\tilde{c}_e \geq 0$.

Algorithm 1: Efficient Pricing Algorithm $\mathcal{O}\left(Q \cdot (|E| + |V| \log |V|)\right)$

for $q = 0, 1, 2, \ldots, Q$ **do**

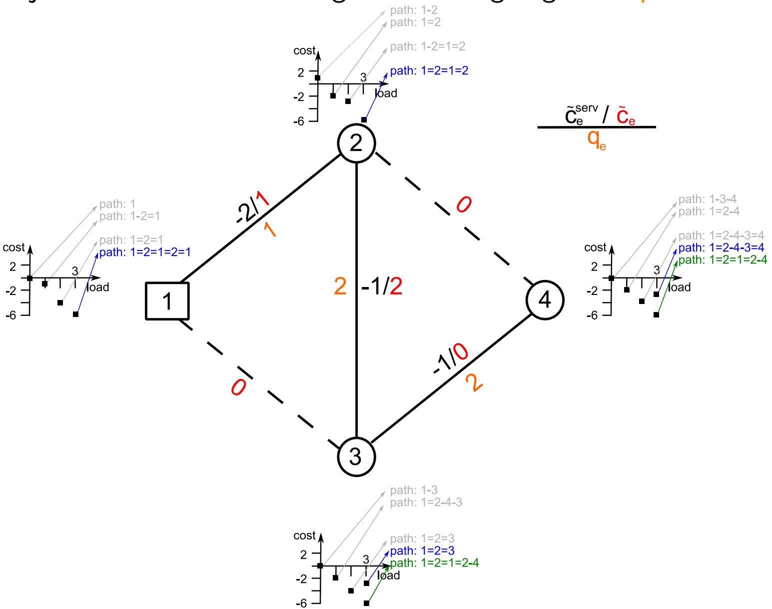
```
// Dijkstra-like extension
//    use rdc c̃_e ≥ 0
//    this creates new labels with identical load q
```

Extend labels (c, q) along deadheading edges and apply dominance algo

```
// Service extension
//    use rdc c̃_e^service
//    this creates only labels with higher load q' > q
```

Extend labels (c, q) along service edges and apply dominance algo

Dijkstra-like Extension along deadheading edges for $q = 3$



Pricing Relaxations

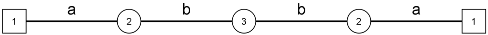
A tour is elementary iff no edge is serviced more than once.

- Nodes may be visited more than once.
- Edges may be traversed more than once, but only serviced once.

Tradeoff: The stronger the relaxation, the better the master program LB, but the harder the pricing problem.

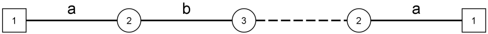
Non-elementary:

feasible:

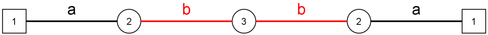


2-loop free:

feasible:



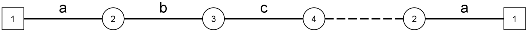
infeasible:



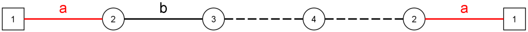
Pricing Relaxations

3-loop free:

feasible:

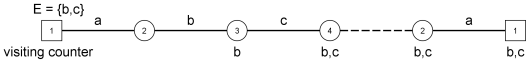


infeasible:

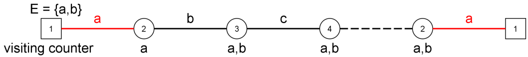


Partial elementary w.r.t. a set $\mathcal{E} \subseteq E_R$:

feasible w.r.t. $\mathcal{E} = \{b, c\}$:

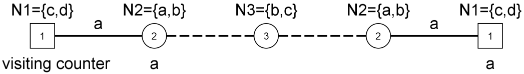


infeasible w.r.t. $\mathcal{E} = \{a, b\}$:

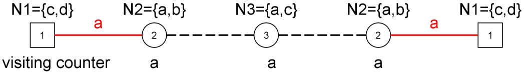


NG-route:

feasible:



infeasible:



Pricing Relaxations

Possible Relaxations (elementary problem is strongly $\mathcal{NP}$-hard):

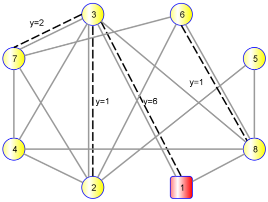
Relaxation	Efficiency Worst Case	Who?
Non-elementary	$\mathcal{O}\left(Q(\lvert E\rvert + \lvert V\rvert \log \lvert V\rvert)\right)$	Letchford and Oukil (2009)
2-loop free = 1-cycle free	Factor 2	based on Houck *et al.* (1980) and Benavent *et al.* (1992)
k-loop free for $k \geq 3$	Factor $k! \cdot (k-1)!$	based on Irnich and Villeneuve (2006)
partial elementarity w.r.t. $\mathcal{E} \subset E_R$	Factor $2^{\lvert\mathcal{E}\rvert}$	Desaulniers *et al.* (2008)
NG-route relaxation w.r.t. $N_i \subset E_R, i \in V$	Factor $2^{\max_i \lvert N_i\rvert}$	Baldacci *et al.* (2009)

Branching

Hierarchical branching scheme with **3 Types of Branching Rules:**

1. Branching on node degrees
   → select node with non-even node degree
2. Branching on edge flows
   → select node with fractional edge flow

Integral x and y variables do **not** guarantee integral route variables λ



Example: '=' is service and '–' is deadheading

Tour 1 (1=3=8=6–8=3–1)	$\lambda_1 = \frac{1}{3}$
Tour 2 (1=8=6–8=5=2=7–3–1)	$\lambda_2 = \frac{1}{3}$
Tour 3 (1–3–2=4=7=3–1)	$\lambda_3 = 1$
Tour 4 (1–3=4=8=3–1)	$\lambda_4 = \frac{1}{3}$
Tour 5 (1–3=2=6=7–3–1)	$\lambda_5 = 1$
Tour 6 (1=3–7=2=5=8=1)	$\lambda_6 = \frac{2}{3}$
Tour 7 (1–3=4=8=6–8=4=3–1)	$\lambda_7 = \frac{1}{3}$

⇒ 3rd class of branching rule is required

3. Branching on followers and non-followers

Branching on Follower Information

Example (cont'd): '=' is service and '–' is deadheading

Tour 1 (1=3=8=6–8=3–1) $\lambda_1 = \frac{1}{3}$
Tour 2 (1=8=6–8=5=2=7–3–1) $\lambda_2 = \frac{1}{3}$
Tour 3 (1–3–2=4=7=3–1) $\lambda_3 = 1$
Tour 4 (1–3=4=8=3–1) $\lambda_4 = \frac{1}{3}$
Tour 5 (1–3=2=6=7–3–1) $\lambda_5 = 1$
Tour 6 (1=3–7=2=5=8=1) $\lambda_6 = \frac{2}{3}$
Tour 7 (1–3=4=8=6–8=4=3–1) $\lambda_7 = \frac{1}{3}$

Consider edges $e = \{1, 3\}$ and $e' = \{3, 8\}$.

→ in $\frac{1}{3}$ of the cases e and e' are direct followers (Tour 1)

→ in $\frac{2}{3}$ of the cases e and e' are not direct followers (Tour 6)

Branching on Follower Information

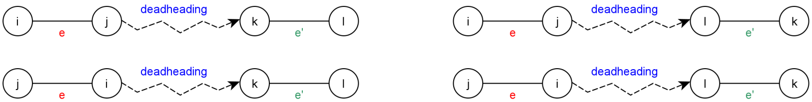
Branching on (non-)followers can be handled by **Network Modifications**:

- Deletions and/or additions of required edges
- Basic structure of the network remains unchanged

Let e and e' be two edges with fractional follower information.

Non-Follower branch: Forbid the consecutive service of these two edges
$\rightarrow$ **Associate same task** with e and e' and perform **2-loop elimination**

Follower branch: Enforce consecutive service
$\rightarrow$ Replace e and e' by **four new edges** modeling consecutive service



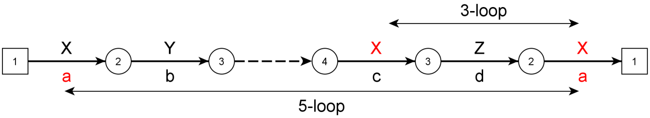
Branching scheme needs pricing problem relaxation that is able to handle two sets of tasks:

Tasks $\mathcal{T}^B$ for branching (2-loop-free tours)

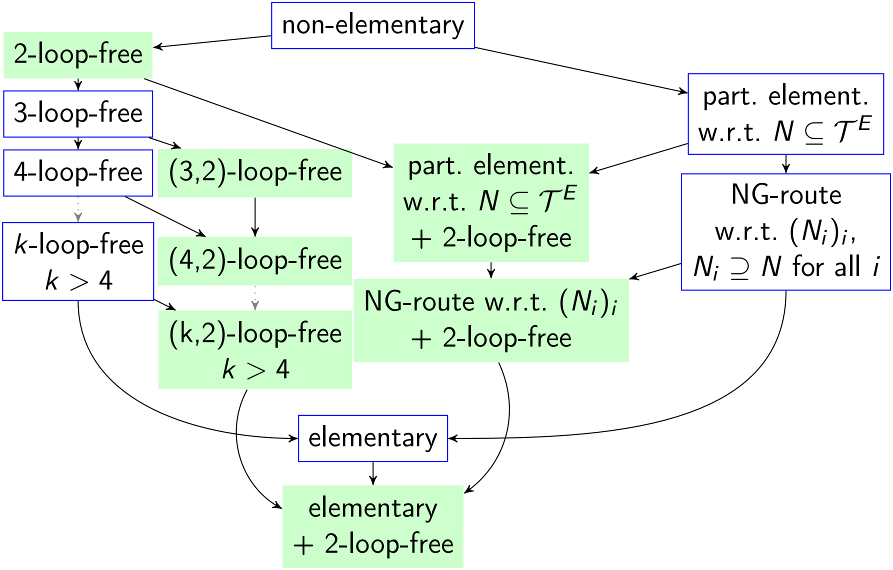
Tasks $\mathcal{T}^E$ for (approximating) elementary routes
(*k*-loop free, partial elementary, NG-route)

Example:

$\mathcal{T}^B = \{X, Y, Z, \dots\}$
$\mathcal{T}^E = \{a, b, c, \dots\}$



Branching and Pricing Relaxations



Note: weaker → stronger

Theoretical Results for $(k, 2)$-Loop Free Pricing

Possible Relaxations (elementary problem is strongly $\mathcal{NP}$-hard):

Relaxation	Efficiency		Who?
	Relaxation $\mathcal{R}$	Relaxation $(\mathcal{R}, 2)$	
Non-elementary	$\mathcal{O}(Q(\lvert E\rvert + \lvert V\rvert \log \lvert V\rvert))$	-	Letchford and Oukil (2009)
2-loop free $=$ 1-cycle free	Factor 2	Factor 2	based on Houck *et al.* (1980 and Benavent *et al.* (1992)
k-loop free for $k \geq 3$	Factor $k! \cdot (k-1)!$	Factor $(k+1)$	based on Irnich and Villeneuve (2006)
partial elementarity w.r.t. $\mathcal{E} \subset E_R$	Factor $2^{\lvert\mathcal{E}\rvert}$	Factor 2	Desaulniers *et al.* (2008)
NG-route relaxation w.r.t. $N_i \subset E_R, i \in V$	Factor $2^{\max_i \lvert N_i \rvert}$	Factor 2	Baldacci *et al.* (2009)

Acceleration Techniques

Heuristic Pricing:

- Reduced Networks (Elimination of Nodes and Arcs)
- Stronger Dominance
- Scaling
- Metaheuristics
- ...

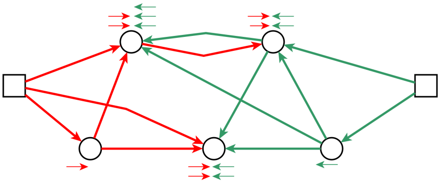
Acceleration of Exact Pricing Algorithms:

- Bidirectional Pricing (Righini and Salani, 2006)
- Bounding (Functions) (Baldacci *et al.*, 2011)

Bidirectional Pricing

Bidirectional Pricing (Righini and Salani, 2006):

1. Propagate FW up to "middle"
2. Propagate BW up to "middle"
3. Merge FW and BW labels

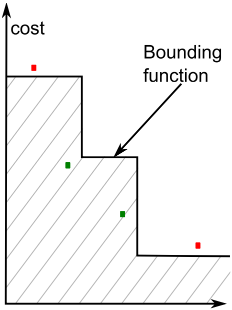


Some Findings:

- In undirected networks there is no difference between forward and backward labels
  - $\Rightarrow$ we save approx. factor 2 in label extension
  - $\Rightarrow$ overall factor 1.5–2; better for $(k, 2)$-loop relaxation
- 2-loop: Simple Merge in $\mathcal{O}(Q)$
  - $\Rightarrow$ guarantees identification of a minimum rdc path
  - $\Rightarrow$ but not all Pareto-optimal paths
- NG-route: Standard half-way test not applicable

Bounding

Bounding is possible with any relaxation of the pricing problem (Baldacci *et al.*, 2011):



Bounding function is computed using (backward) labeling with a proper relaxation

Pricing with	Relaxation for Bounding
2-loop	non-elementary scaled instance
3-loop	non-elementary 2-loop scaled instance
4-loop	non-elementary 2-loop or 3-loop scaled instance
NG-route + 2-loop	2-loop NG-route with subset of neighbors

Note: No Half-way Stop!

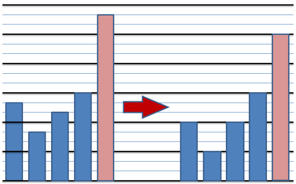
Scaling

Scaling can be used as a . . .

Relaxation:

Rounded down demand and capacity to next integer multiple of a factor f

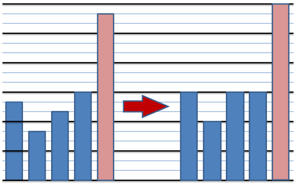
Such a relaxation can be used for bounding.



Restriction:

Rounded up demand and capacity to next integer multiple of a factor f

Can be used as a pricing heuristic.



Computational Setup:

- Computation times do not include Phase 1 (cutting), Time limit $TL = 4$ hours
- Best node first is node selection rule for B&B
- All Pricing Problems with bounding and solved bidirectionally
- NG-neighborhoods are generated dynamically based on iteratively solving the branch-and-price root node
  - all neighborhoods $N_i := \emptyset$ at start
  - select cycle $C = (i_1, i_2, \dots, i_k, i_1)$ with maximum flow
  - first and last serviced/required edges are identical $e = \{i_1, i_2\} = \{i_k, i_1\}$
  - task e is added to neighborhoods $N_{i_1}, N_{i_2}, \dots, N_{i_k}$
  - stop if bound on neighborhood size is exceeded

Best known Bounds

Table : Best Known Bounds for the egl Instances

instance	lb_{best}	computed by	ub_{best}	computed by	*opt*	proved by
egl-e2-b			6317	Brandão and Eglese (2008)	6317	own
egl-e3-b	7744	own	7775	Polacek *et al.* (2008)		
egl-e3-c	10244	Bartolini *et al.* (2012)	10292	Polacek *et al.* (2008)		
egl-e4-a	6408	Bode and Irnich (2012)	6444	Santos *et al.* (2010)		
egl-e4-b	8935	Bartolini *et al.* (2012)	8961	Bartolini *et al.* (2012)		
egl-e4-c	11512	own	11529	own		
egl-s2-a	9825	Bartolini *et al.* (2012)	9884	Santos *et al.* (2010)		
egl-s2-b	13017	Bartolini *et al.* (2012)	13100	Brandão and Eglese (2008)		
egl-s3-a	10165	own	10220	Santos *et al.* (2010)		
egl-s3-b	13648	Bartolini *et al.* (2012)	13682	Polacek *et al.* (2008)		
egl-s4-a	12153	own	12268	Santos *et al.* (2010)		
egl-s4-b	16113	own	16283	Fu *et al.* (2010)		
egl-s4-c	20430	Bartolini *et al.* (2012)	20481	Bartolini *et al.* (2012)		
egl-g1-a	976907	own	1049708	Martinelli *et al.* (2011)		
egl-g1-b	1093884	own	1140692	Martinelli *et al.* (2011)		
egl-g1-c	1212151	own	1282270	Martinelli *et al.* (2011)		
egl-g1-d	1341918	own	1420126	Martinelli *et al.* (2011)		
egl-g1-e	1482176	own	1583133	Martinelli *et al.* (2011)		
egl-g2-a	1067262	own	1129229	Martinelli *et al.* (2011)		
egl-g2-b	1185221	own	1255907	Martinelli *et al.* (2011)		
egl-g2-c	1311339	own	1417145	Martinelli *et al.* (2011)		
egl-g2-d	1446680	own	1516103	Martinelli *et al.* (2011)		
egl-g2-e	1581459	own	1701681	Martinelli *et al.* (2011)		

Best known Bounds

Table : Best Known Bounds for the bmcv Instances, Subsets C and E

instance	lb_{best}	computed by	ub_{best}	computed by	*opt*	proved by
C01	4145	own	4150	Beullens *et al.* (2003)		
C04			3510	Beullens *et al.* (2003)	3510	own
C09	5245	own	5260	Brandão and Eglese (2008)		
C11	4615	own	4630	Mei *et al.* (2009)		
C12	4235	own	4240	Beullens *et al.* (2003)		
C15	4920	own	4940	Beullens *et al.* (2003)		
C18	5580	Bartolini *et al.* (2012)	5620	Santos *et al.* (2010)		
C19			3115	Beullens *et al.* (2003)	3115	own
C21			3970	Beullens *et al.* (2003)	3970	own
C23	4075	own	4085	Beullens *et al.* (2003)		
C24			3400	Beullens *et al.* (2003)	3400	own
E01	4900	own	4910	Brandão and Eglese (2008)		
E09	5805	own	5820	Tang *et al.* (2009)		
E11					4650	own
E15	4200	own	4205	Santos *et al.* (2010)		
E16			3775	Beullens *et al.* (2003)	3775	own
E18			3835	Beullens *et al.* (2003)	3835	own
E19			3235	Beullens *et al.* (2003)	3235	own
E20			2825	Beullens *et al.* (2003)	2825	own
E23			3710	Beullens *et al.* (2003)	3710	own
E24			4020	Beullens *et al.* (2003)	4020	own

Best known Bounds

Table : Best Known Bounds for the bmcv Instances, Subsets D and F

instance	lb_{best}	computed by	ub_{best}	computed by	*opt*	proved by
D08			3045	Beullens *et al.* (2003)	3045	own
D14			3280	Beullens *et al.* (2003)	3280	own
D19			2400	Beullens *et al.* (2003)	2400	own
D21	3005	own	3050	Beullens *et al.* (2003)		
D23			3130	Beullens *et al.* (2003)	3130	own
D24	2705	own	2710	Beullens *et al.* (2003)		
F04			3485	Beullens *et al.* (2003)	3485	own
F08			3705	Beullens *et al.* (2003)	3705	own
F12			3395	Beullens *et al.* (2003)	3395	own
F18	3065	Bartolini *et al.* (2012)	3075	Beullens *et al.* (2003)		
F19	2515	own	2525	Beullens *et al.* (2003)		
F23			3005	Beullens *et al.* (2003)	3005	own

Conclusions and Outlook

Conclusions:

- "Easy" instances: best to be solved with 2-loop or 3-loop relaxation
- "Hard" instances: often best LBs with NG-route relaxation
  - dynamic generation of neighborhoods $N_i \subseteq E_R$
  - maximum neighborhood size 5, 6 or 7 works best
- Speedup by combining Bounding and Bidirectional Pricing gives a speedup by
  - 4-loop: approx. factor 8.0
  - NG-route/3-loop: approx. factor 4
- Partial Elementary is not competitive with NG-route relaxation

Outlook:

1. Detailed experiments with Strong Branching necessary
2. Integration Subset-row inequalities and other non-robust Cuts

Thank you for coming!

Questions?!

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Network Modification

Modifications become more intricate if several follower and non-follower constraints are active.

Example: Active follower decisions $F = \{(e_1, e_2), (e_3, e_4)\}$ and non-follower decisions $N = \{(e_1, e_3)\}$.

F and N partition E_R and one partition is $\{e_1, e_2, e_3, e_4\}$.

The set of all possible subsequences s is

$$\{(e_1, e_2), (e_3, e_4), (e_1, e_2, e_3, e_4), (e_1, e_2, e_4, e_3), (e_2, e_1, e_4, e_3)\}$$

These subsequences

- are represented by 4 edges each
- cost of a subsequence can be determined solving a DP