A quantitative analysis of (in)stability in the morning commute problem

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1 Introduction

Since its introduction by Vickrey (1969), the problem of departure time choice in the morning commute has been widely studied. Existence and uniqueness of equilibrium were established under various specifications (Smith, 1984a; Daganzo, 1985; Lindsey, 2004), many interesting properties were identified (Arnott et al., 1990), and it has now become a component of many more complex models, involving for instance various modes and public policies.

However, the few existing results reported related to equilibrium stability are disturbing. In fact, almost all the works assuming fully rational revision protocols (i.e. the protocols by which users update their departure time choice given the current costs of all alternatives) found the equilibrium to be unstable, such that even with constant inelastic demand, the system perpetually oscillates in some region of the feasible space (de Palma, 2000; Iryo, 2008; Bressan et al., 2012; Guo et al., 2018).

Such an observation naturally raises multiple questions. From a practical point of view, the most important question is perhaps whether the system remains sufficiently close to its equilibrium for instability to be simply ignored. While this question can hardly be answered given the current state of knowledge, this paper aims at improving our understanding of departure time choice as a dynamic process by identifying the effect on stability of the three ingredients of the problem: the congestion mechanism, the revision protocol and the schedule preferences. Unlike previous works on stability that considered specific combinations of those, this paper provides some qualitative results for the bottleneck model that are valid for large classes of revision protocols and schedule preferences. Besides, we extend our analysis by considering also other congestion mechanisms. We treat in particular the case of an isotropic region, modeled by a Macroscopic (or Network) Fundamental Diagram (MFD). This parallel treatment of the bottleneck model and of the isotropic region suggests that the spatial configuration of the network is perhaps the most crucial ingredient in stability analysis.

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Such a general qualitative analysis is made possible by the existence of strong theoretical results that were established for monotonic cost functions, in particular in the fields of route choice (Smith, 1979; Dafermos, 1980; Smith, 1984b; Smith and Ghali, 1990; Mounce, 2006) and evolutionary game theory (Sandholm, 2015). This paper shows that by specifying the schedule preferences appropriately, the same theoretical results can actually be applied to the departure time choice problem.

2 Methodological background

We consider a single population of homogeneous users, who choose departure times within some choice set. In the literature on the morning commute, the set of departure times is typically the set of real numbers, or some continuous interval. However, since most of the existing stability results were only proven for discrete choice sets (e.g. a finite number of routes), we also choose this setting for this brief introduction. For a more detailed introduction, the reader can refer to Sandholm (2015).

The set of possible populations states is thus $X = \{(x_1, ...x_n) \in \mathbb{R}_+^n, \sum_{i=1}^n x_i = 1\}$, where x_i denotes the proportion of users choosing strategy i. A population game is characterized by a continuous cost function $C : X \to \mathbb{R}^n$, where $C_i(x)$ represents the cost of strategy i when the population plays according to x.[1]

Unlike classical game theory, population games are generally studied dynamically. Agents receive sporadically the opportunity to update their decision and do so by applying a so-called "revision protocol". In the present paper, we focus on reactive protocols of the form $\rho : X \times \mathbb{R}^n \to \mathbb{R}_+^{n\times n}$, which map population states $x \in X$ and their corresponding cost vectors $c \in \mathbb{R}^n$ to matrices of switch rates ρ_{ij}. Assuming that the revision protocol is Lipschitz continuous, it defines a "deterministic evolutionary dynamic", i.e. a map that assigns to each continuous cost function $C : X \to \mathbb{R}^n$ an ordinary differential equation $\dot{x} = V(x)$, where $V_i(x) = \sum_{j=1}^n x_j \rho_{ji}(x, C(x)) - x_i \sum_{j=1}^n \rho_{ij}(x, C(x))$.

Many revision protocols have been proposed in the literature (Sandholm, 2015). For brevity, we only present here the one introduced by Smith (1984b). It is defined by $\rho_{ij}(x, c) = [c_i - c_j]_+$ (where $\forall y \in \mathbb{R}, [y]_+ = \max(0, y)$) and it leads to the following continuous dynamic:

$$\dot{x_i} = \sum_{j=1}^n x_j \left[C_j(x) - C_i(x)\right]_+ - x_i \sum_{j=1}^n \left[C_i(x) - C_j(x)\right]_+ . \tag{1}$$

The Smith dynamic can be interpreted as follows: every user regularly revises her decision by comparing her current strategy against a randomly selected alternative. If the alternative provides a smaller cost, the user adopts it with a probability that is proportional to the cost difference. Otherwise, she retains her current strategy. We are now ready to introduce the tools that are relevant for stability analysis.

Definition 1. *A cost function $C : X \to \mathbb{R}^n$ is monotonic if $\langle y - x, C(y) - C(x)\rangle \geq 0$ for all $x, y \in X$.*

An important consequence of monotonicity is that the set of Nash equilibria is convex. If in addition the cost function is strictly monotonic at some Nash equilibrium x^* (i.e. if $\langle y - x^*, C(y) - C(x^*)\rangle > 0$ for all $y \in X \backslash \{x^*\}$), then this Nash equilibrium is unique. Provided that the cost function is continuously differentiable, monotonicity also guarantees that the set of Nash equilibria is globally attracting for multiple dynamics, including those of Smith (1984b) and many others based on rational revision protocols (Hofbauer and Sandholm, 2009).

[1] Payoff functions are actually more common in game theory, but cost functions are more common in transportation.

3 Bottleneck model

We consider schedule preferences that can be expressed as cost functions of the form $C(t_d, t_a) = -H(t_d) - W(t_a)$, where t_d and t_a are the times of departure (from home) and arrival (at work) and $H(t_d) = \int_0^{t_d} h(t_d)\,\mathrm{d}t$ and $W(t_a) = \int_{t_a}^0 w(t_a)\,\mathrm{d}t$. The marginal utility rates at home (h) and at work (w) are assumed to be positive everywhere. To ensure the existence and uniqueness of equilibrium, we assume that there exists t^* such that for all $t < t^*$, $h(t) > w(t)$ and for all $t > t^*$, $h(t) < w(t)$. Besides realism and empirical support (Tseng and Verhoef, 2008), such a specification of schedule preferences also allows for particularly simple yet general results in terms of monotonicity.

We now introduce a result of Mounce (2006), established in the context of dynamic user equilibrium (without choice of departure time).

Theorem 1 (Mounce (2006))**.** *The bottleneck delay function is a monotonic function of the flow into the bottleneck if and only if the bottleneck capacity is non-decreasing with respect to time.*

In this context, a bottleneck delay function is called monotonic if $\int_0^1 (d_x(t) - d_y(t))(x(t) - y(t))\,\mathrm{d}t \geq 0$ for all inflow functions x and y, where $d_x(t)$ denotes the delay experienced by a user arriving at the bottleneck at time t when the inflow profile is given by the function $x(t)$ (inflow functions are assumed to be real-valued, non-negative, measurable and essentially bounded on $[0, 1]$).

Theorem 1 can actually be transposed to the case with departure time choice by defining similarly $c_x(t) = -H(t) - W(t + d_x(t))$, the cost experienced by a user arriving at the bottleneck at time t given the inflow function x and by applying the change of variable $u = \int_0^t w(\tau)\,\mathrm{d}\tau$. In that space, given some departure time, every additional interval of travel δu has the same value and the bottleneck capacity, which becomes u-dependent, is given by $\hat{s}(u) = s(t(u))/w(t(u))$ (where $t(u)$ is obtained by making the inverse change of variable). Thus, we obtain the following corollary:

Corollary 1. *With a bottleneck of constant capacity, the bottleneck cost function is a monotonic function of the flow into the bottleneck if and only if the marginal utility function at work w is a non-increasing function of time.*

Since the marginal utility rate w is expected to be an increasing function of time (Tseng and Verhoef, 2008), Corollary 1 implies that the cost function associated with the bottleneck is not monotonic.

It is of practical interest to notice that pricing can be used not only to decentralize the social optimum, but also to stabilize it. To show this, let us denote $\bar{C}$ the maximum cost under a socially optimal inflow function and consider the following toll function of the passage time at the bottleneck:

$$\$(t) = \max(H(t) + W(t) + \bar{C}, 0) \tag{2}$$

Since the social optimum does not involve any queueing, the bottleneck should be used at capacity at all times such that $-\bar{C} < H(t) + W(t)$. At such times, the generalized cost becomes $C = -H(t_d) - W(t_a) + \bar{C} + H(t_a) + W(t_a) = -H(t_d) + \bar{C} + H(t_a) = -H(t_d) + \bar{C} - \int_{t_a}^0 h(t)\,\mathrm{d}t$. By applying the change of variable $u = \int_0^t h(\tau)\,\mathrm{d}\tau$, we obtain another corollary.

Corollary 2. *With a bottleneck of constant capacity and the toll* (2)*, the bottleneck generalized cost function is a monotonic function of the flow into the bottleneck if and only if the marginal utility function at home h is a non-increasing function of time.*

If the marginal utility rate of being at home does decrease over the course of the morning (as it is intuitively expected and observed empirically by Tseng and Verhoef (2008)), the social optimum obtained with the toll (2) is globally attracting for many rational revision protocols.

4 Isotropic region

The isotropic region offers a fundamentally different model of congestion. With the bottleneck model, travel time only depends on the queue length *at the time the user enters the queue*. In an isotropic region however, travel time depends on the accumulation of vehicles in the network *throughout the trip*. This difference makes many derivations intractable. Consequently, we propose to study the monotonicity of the cost function via two means. First, it is shown that under light congestion, the cost function can be well approximated by a simpler function that is provably monotonic. Second, the local stability of the equilibrium is investigated numerically by relying on the concept of Evolutionarily Stable State, for which other general stability results have been proven (Sandholm, 2010). Together, these results suggest that the equilibrium corresponding to isotropic dynamics is locally stable under light congestion for a broad class of revision protocols but may become unstable under more severe congestion.

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