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MSA Algorithms for solving SUE in Urban Transit Networks

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Abstract

This paper addresses the Stochastic User Equilibrium assignment with mixed pre-trip en-route path choice behavior. This problem is relevant for urban transit networks: when travelers begin their journey they may not completely know the status of service, say bus arrivals at stops, and may only choose a travelling strategy at origin. Under mild assumptions such strategies can be modelled as hyperpaths, thus user choice behavior can be modelled through Random Utility Models applied to the set of (elementary) hyperpaths, such as Probit and Gammit; these models allow for general co-variance matrices but their application requires Montecarlo techniques. Several MSA-based algorithms have been presented and compared on a simple network using both Mersenne Twister pseudo-random numbers and Sobol quasi-random numbers.

Keywords *G E Cantarella, SUE, transit networks, hyperpaths, MSA, PSRG*

1. INTRODUCTION

The User Equilibrium (UE) paradigm was introduced by Wardrop (1952) for deterministic path choice behaviour. Daganzo and Sheffi (1977) extended the User Equilibrium to path choice behaviour described through a random utility model, calling it *Stochastic User Equilibrium* (SUE). The user equilibrium problem was formulated as fixed-point problem based on the inverse cost function by Daganzo (1983); Cantarella (1997) provides a general fixed-point modelling framework, which does not need the inverse cost function.

The Method of Successive Averages (MSA) is extensively used for specifying solution algorithms for SUE suitable for large scale applications (see Cantarella, 1997, for more details and convergence conditions). Recently Di Gangi at al. (2015) proposed and discussed several MSA-based algorithms for solving SUE based on path choice models with general covariance matrices but requiring Montecarlo simulation for their solution, such as Probit (Daganzo, 1983) or Gammit (Cantarella and Binetti, 2002).

The equilibrium *assignment to a transit network* can be based on either of two different approaches to path choice behaviour modelling.

- Mixed pre-trip and en-route strategies, typical for high frequency systems; pre-trip a user chooses a travel strategy, while en-route the user chooses a line to board at each stop, leading to the so-called *frequency-based assignment*, will be addressed in this paper.
- Fully pre-trip, typically for low frequency systems; pre-trip a user chooses a sequence of runs leading to the so-called scheduled-based assignment; this is often the case for extra-urban trips, on trains, air flights, … and will be addressed in a future paper.

In early studies the user strategy was restricted to the case of overlapping lines with equivalent speeds, which in turn were replaced by a common line with a frequency equal to the sum of single lines frequencies. Then, in successive studies the case of overlapping transit lines with different speeds was addressed by introducing the concept of optimal strategy (Spiess, 1984), which may be modelled through a *hyperpath* under mild assumptions (Nguyen and Pallottino, 1988), leading to methods of deterministic user equilibrium assignment to a congested transit network (Wu and Florian, 1993; Wu *et al.*, 1994). The general fixed-point approach reported in Cantarella (1997) for SUE was further analysed for high frequency transit systems by Cantarella and Vitetta (2001).

In this paper we will extend SUE MSA-based algorithms described in Di Gangi et al. (2015) to deal with urban transit networks, where routing choice alternatives can be modeled through hyperpaths. Several probabilistic choice models, corresponding to different distributions for perceived cost, are analysed. In section 2 models for SUE frequency-based assignment are reviewed, extending notations of the above cited Di Gangi et al. (2015), and in section 3 SUE assignment solution algorithms are described. In section 4 implementation issues and some numeric results for a test system are discussed. Some conclusions and indications for further research are reported in the last section.

2. SUE ASSIGNMENT MODELS

Models for traffic assignment to transportation networks simulate how demand and supply interact in transportation systems. These models enable the calculation of performance measures and user flows for each supply element, resulting from origin-destination demand flows, path choice behaviour, and the reciprocal interactions between supply and demand. Assignment models combine two sub-models: the supply model and the demand model. In this section fixed-point models for SUE frequency-based assignment are briefly reviewed (for more details see Cantarella, 1997).

In the following user travelling between the same origin-destination pair with common behavioural features are grouped into a user class i, with a *demand flow* $d_i \geq 0$, a (non-empty and finite) set of (elementary) available paths K_i and a corresponding set of hyperpaths.

2.1 SUPPLY MODEL

Transportation supply models express how user behaviour affects network performances. They are usually based on congested network models, that is a graph $G(N, A)$ with a transportation cost c_a and a flow f_a associated to each arc a in set A. Let

$\mathbf{B}_i$ be the *arc-path incidence matrix* for user class i with entries $b_{ak} = 1$ if arc a belongs to path k, $b_{ak} = 0$ otherwise;
$\mathbf{h}_i \geq \mathbf{0}$ be the *path flow vector* for user class i, with entries h_k, $k \in K_i$;
$\mathbf{f} \geq \mathbf{0}$ be the *arc flow vector*, with entries f_a, $a \in A$;
$\mathbf{c}$ be the *arc cost vector*, assumed below with non negative entries $c_a \geq 0$, $a \in A$;
$\mathbf{g}_i$ be the *path cost vector* for user class i, with entries w_k, $k \in K_i$.

The following three equations completely describe the transportation supply :

$$\mathbf{f} = \Sigma_i \, \mathbf{B}_i \, \mathbf{h}_i \tag{1}$$
$$\mathbf{c} = \mathbf{c}(\mathbf{f}) \tag{2}$$
$$\mathbf{g}_i = \mathbf{B}_i^{\mathrm{T}} \, \mathbf{c} \quad \forall i \tag{3}$$

The function in equation (2) is called the *arc cost function*. A journey starts from an origin and arrive in a destination traversing arcs of different kinds: access/egress or pedestrian, waiting, boarding, alighting, on-board. The arc cost functions depend on the arc type (Fig. 1). Examples are given in section 4.

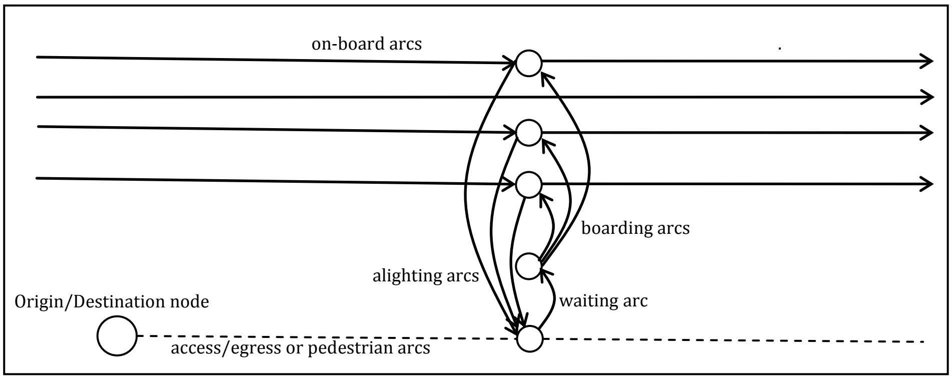


Figure 1. Arcs and nodes modelling a bus stop

2.2 DEMAND MODEL

At the origin a user chooses a strategy, defined by a set of diversion nodes and a set of boarding options, say lines, at each diversion node – PRE-TRIP CHOICE BEHAVIOUR; each strategy is a set of partially overlapping paths (possibly a single path), which may be represented by a hyperpath (for more details see Nguyen and Pallottino, 1988) under mild assumptions. Then, at each diversion node, that is each waiting node, the user chooses the line to board among those considered in the pre-trip strategy – EN-ROUTE CHOICE BEHAVIOUR; the sequence of en-route choices defines the path actually followed to destination, among all paths belonging to the hyperpath chosen at origin.

EN-ROUTE CHOICE BEHAVIOUR MODELLING. Let

$\eta_{aj} \in [0,1]$ be the *en-route diversion probability* for arc a and hyperpath j, such that:

$\eta_{aj} \in]0,1]$ if $a \in j$ and it is a boarding arc (with sum equal to 1 for all arcs exiting from the same waiting node);
$\eta_{aj} = 1$ if if $a \in j$ and it is not a boarding arc;
$\eta_{aj} = 0$ $a \notin j$;

The diversion probability η_{aj} can be modelled according with the adopted rules at a waiting node, the most common being “take the first bus arriving at the stop”. The waiting time can be computed consistently, see section 4 for more detail and some examples. Let

$q_{kj} = \Pi_{a \in k}\, \eta_{aj} \in [0,1]$ be the probability of following path k within hyperpath j, if the en-route choices at diversion nodes are stochastically independent. It is worth noting that a path may well belong to several hyperpaths. By definition $q_{kj} > 0 \Leftrightarrow k \in j$; $q_{kj} = 0 \Leftrightarrow k \notin j$. Let

Q be the *path-hyperpath probability matrix* for user class i with entries q_{kj}.

PRE-TRIP CHOICE BEHAVIOUR MODELLING. Let

$\boldsymbol{U}_i$ be the hyperpath perceived utility vector for user class i, modelled as a random vector according to Random Utility Theory;
$\mathbf{v}_i = E[\boldsymbol{U}_i]$ be the expected value of $\boldsymbol{U}_i$ or the hyperpath systematic utility vector for user class i;
$\mathbf{p}_i \geq \mathbf{0}$, with $\mathbf{1}^T \mathbf{p}_i = 1$, be the hyperpath choice vector for user class i.

The systematic utility is given by the sum of some generic and additive cost attributes, such as access/egress or pedestrian, boarding, alighting, on-board times, which do not depend on the hyperpath, and hyperpath specific cost attributes, say waiting times, which depend on the hyperpath, since they depend on the set of boarding options, lines, in the hyperpath. Let

$\mathbf{x}_i^G$ be the vector of hyperpath generic costs;
$\mathbf{x}_i^S$ be the vector of hyperpath specific costs.

The vector of hyperpath generic attributes can be computed from the path costs as:

$$\mathbf{x}_i^G = \mathbf{Q}_i^T \mathbf{g}_i \qquad \forall i \tag{4}$$

and the hyperpath systematic utility vector is given by (omitting weighting coefficients for simplicity’s sake) the *hyperpath utility function*:

$$\mathbf{v}_i = -\mathbf{x}_i^G - \mathbf{x}_i^S \quad \forall i \tag{5}$$

The hyperpath choice probability vector, $\mathbf{p}_i$, is a function of the systematic utility, derived from the random utility theory:

$$\mathbf{p}_i = \mathbf{p}_i(\mathbf{v}_i) \tag{6}$$

If the perceived utility distribution parameters (not considering the mean) do not depend on systematic utility values the *hyperpath choice function* (6) is monotone increasing, with symmetric positive semi-definite Jacobian, with respect to $\mathbf{v}_i$ (Cantarella, 1997).

Examples of the choice function (6) are, apart the well-known Logit, the Probit (Daganzo, 1983) or the Gammit (Cantarella and Binetti, 2002) choice models based on MultiVariate Normal or Gamma distributions of hyperpath perceived utility respectively. These models allow for general structure of the co-variance matrix but cannot be expressed in a closed form, thus MonteCarlo techniques are used for applications (as described in sub-section 3.1). Well established Probit model shows a drawback (as all Logit choice models based on Gumbel distribution) since it allows positive perceived utility values too. Nielsen (1997) explores different ways of reducing this problem with symmetrical truncation of Normal or the use of Log-Normal distribution for perceived arc costs. Sheffi (1985) and Nielsen (1997) suggested the use of Gamma distribution for perceived costs. More recently Gammit choice models based on Gamma distribution have been deeply analysed by Cantarella and Binetti (2002).

For practical purposes to apply the Probit or the Gammit model, hyperpath perceived utilities $\boldsymbol{U}_i$ can be specified through arc perceived costs $\boldsymbol{W}$ as: $\boldsymbol{U}_i = -\mathbf{Q}_i^T \mathbf{B}_i^T \boldsymbol{W}$, expected values of arc perceived costs being the arc costs, $E[\boldsymbol{W}] = \mathbf{c}$. If the arc perceived costs are independently Normal or Gamma distributed with diagonal covariance matrix $\boldsymbol{\Sigma}_\mathbf{W}$, the resulting hyperpath perceived utilities are still Normal or Gamma (for more details see Moschopoulos, 1985), distributed with covariance matrix $\mathbf{Q}_i^T \mathbf{B}_i^T \boldsymbol{\Sigma}_\mathbf{W} \mathbf{B}_i \mathbf{Q}_i$ with non null covariance for each pair of partially overlapping hyperpaths. Specifications of $\mathbf{W}$ are given below. Let

$c_{o,a} > 0$ be the zero-flow cost arc on arc $a \in A$, such that $c_a(\mathbf{f}) \geq c_{o,a}\ \forall\ \mathbf{f}$ [this condition surely occurs for monotone increasing cost functions];

$\tau > 0$ be the dispersion parameter, assumed below less than 1 [for $\tau = 0$ the hyperpath choice behaviour is deterministic]; such that $\mathrm{Var}(W_a) = \tau\ c_{o,a}$, under this assumption the perceived utility distribution parameters (not considering the mean) do not depend on systematic utility values the hyperpath choice function (6).

- for Probit

$$W_a \sim (c_a - c_{o,a}) + \mathrm{Normal}(\mu = c_{o,a}, \sigma = \tau\ c_{o,a}) = c_a + \mathrm{Normal}(\mu = 0, \sigma = \tau\ c_{o,a})$$

- for Gammit

$$W_a \sim (c_a - c_{o,a}) + \mathrm{Gamma}(\mu = c_{o,a}, \sigma = \tau\ c_{o,a}) = (c_a - c_{o,a}) + \mathrm{Gamma}(\alpha = c_{o,a}/\tau, \beta = \tau)$$

where $\beta = \tau$ is common to all arcs.

DEMAND FLOW CONSERVATION. Let

$d_i \geq 0$ be the demand flow vector for user class I, assumed fixed in this paper;
$\mathbf{y}_i \geq \mathbf{0}$ be the hyperpath flow vector for user class i.

The hyperpath flow vector is given by:

$$\mathbf{y}_i = d_i\, \mathbf{p}_i \tag{7}$$

and the path flow vector by:

$$\mathbf{h}_i = \mathbf{Q}_i\, \mathbf{y}_i \tag{8}$$

2.3 DEMAND - SUPPLY INTERACTION MODEL

The (stochastic) *arc flow function* with constant demand is obtained by combining supply model equations (1) and (3) with the demand model as described by equations (4-8):

$$\mathbf{f}(\mathbf{c};\, \mathbf{d}) \equiv \sum_i d_i\, \mathbf{B}_i\, \mathbf{Q}_i\, \mathbf{p}_i(-\mathbf{Q}_i^T\, \mathbf{B}_i^T \mathbf{c}) \qquad (9)$$

The (stochastic) *arc flow function* specifies the so-called stochastic network loading (SNL) that is the assignment to uncongested networks. With the generally adopted random utility, the arc flow function is continuous with continuous first partial derivatives with respect to the arc cost vector, and under mild assumptions monotone non increasing with symmetric negative semi-definite Jacobian. This function is also useful to specify equilibrium models, as shown below.

The multi-user equilibrium assignment to a transportation network with constant demand can be specified by the system of non-linear (vector) equations (1-8); it can be easily recognized that the number of equations is equal to the number of unknowns.

In order to make the analysis of the model easier, it is common practice to combine all equations into one single (vector) equation leading to a fixed-point model with respect to arc flows. The same model is obtained by combining the arc flow function (9) with the arc cost function (2):

$$\mathbf{f}^* = \mathbf{f}(\mathbf{c}(\mathbf{f}^*) \qquad (10)$$

If the network is connected and if all the involved functions (arc cost, hyperpath utility, hyperpath choice) are continuous, the existence of a solution can easily be proved. If the arc flow function is monotone non-increasing, and if arc cost function is monotone strictly increasing at most one solution (weak uniqueness) can easily be proved (weaker conditions are also available). Existence or uniqueness of the arc flow vector also guarantees existence or uniqueness of the arc cost vector, as well as of the path/hyperpath flow and cost vectors (for more details on the above issues see Cantarella, 1997).

3. SOLUTION ALGORITHMS

The fixed-point problem (10) is usually solved by iteratively computing the arc cost function (2) and the arc flow function (9) within the averaging scheme of the MSA. Below first it is described how the arc flow function can be computed, without explicit enumeration of paths and hyperpaths; then, several MSA-based algorithms for solving the fixed-point model (10) are reported.

3.1 COMPUTING THE ARC FLOW FUNCTION

When a closed form is not available for the choice behaviour model (an unbiased estimate of) the arc flow function (9) can be computed through a Montecarlo technique (Burrell, 1968; Sheffi, 1985) by successive averaging several demand loading to the shortest hyperpath. As already said, to apply this approach the hyperpath perceived utility distribution should be based on independently distributed arc perceived costs as described in sub-section 2.2.

The main steps of the procedure are:

k = 0
$\mathbf{f}_{SNL}^{0}$:= **0**
repeat:
  k += 1
  c* *= pseudo-realisation of perceived costs*
  $\mathbf{f}^{k}_{AON}$ = *shortest hyperpath loading with costs* **c***
  $\mathbf{f}^{k}_{SNL}$= [(k-1) $\mathbf{f}^{k-1}_{SNL}$ + $\mathbf{f}^{k}_{AON}$] / k

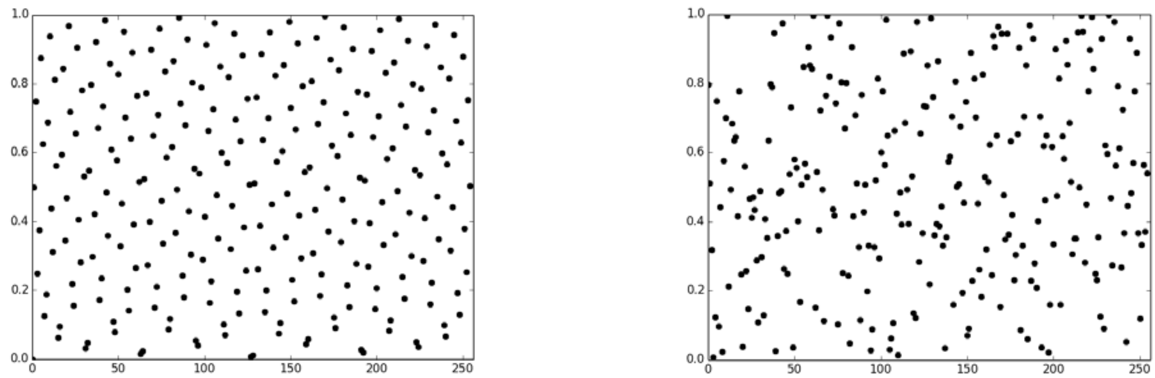
until ($\mathbf{f}^{k-1}_{SNL} \cong \mathbf{f}^{k}_{SNL}$) or a prefixed number of iterations is reached
[It seems worth stressing that the Montecarlo technique is used as a numerical tool to compute the hyperpath choice probabilities, or better the corresponding arc flows, but the whole model is still macroscopic. Thus, no kind of microscopic discrete simulation is actually carried-out.]

PSEUDO-REALISATION OF PERCEIVED COSTS.
Details of the realisation of the arc perceived costs are given below. Let

$\Phi_a(\cdot)$ be the distribution function of the arc perceived cost arc on arc $a \in A$, as in the examples described in sub-section 2.2; if this function cannot be expressed in a closed form (as for Normal) or has a complicated closed form (as for Gamma), approximation functions are used (Abramowiz and Stegun, 1970).

Hence given a random number r, uniformly distributed over interval [0, 1], a realisation of the arc perceived cost on arc a is given by $\Phi_a^{-1}(r)$.

Value r can actually be generated with a Pseudo-Random Number Generator (PRNG). In literature various procedures are proposed for PRNG (i.e. Sánchez *et al.*, 2005; Wichmann and Hill, 2006; Marchi *et al.*, 2009); the aim is, in general, to cover the space in the most uniform way possible. In this paper, two approaches for PRNG have been considered: the Mersenne Twister (Matsumoto and Nishimura, 1998) pseudo-numbers and the Sobol sequences of (often called) quasi-random numbers (Sobol, 1967). Figure 2 shows 256 Sobol a) and Mersenne-Twister b) numbers. In general, the Sobol numbers cover the space more evenly than the Mersenne Twister numbers. For this reason, the convergence of algorithms using these sequences is likely faster.



a) b)

Figure 2. Realizations or random numbers: Sobol (a) versus Mersenne Twister (b)

SHORTEST HYPERPATH LOADING
Given a vector of arc costs the arc flows resulting from a shortest hyperpath loading can be computed through a generalisation of backwards label-setting shortest paths algorithms (Nguyen and Pallottino, 1988) and the forward loading of demand flows to the shortest hyperpath tree to each destination, without explicit enumeration of paths and hyperpaths.

3.2 SOLVING THE FIXED-POINT MODEL

The solution of the fixed-point model (10) can be carried out with a MSA algorithm where arc flows or costs are updated at each iteration, leading to the Flow Averaging (MSA-FA) or the

Cost Averaging (MSA-CA) algorithm (see Cantarella, 1997 for convergence analysis and references); their steps are showed in Figures 3 and 4, where:

k is the iteration counter;
α^k is the step size.

$$\mathbf{c}^k = \mathbf{c}(\mathbf{f}^{k-1}),$$
$$\mathbf{fs}^k = \mathbf{f}(\mathbf{c}^k),$$
$$\mathbf{f}^k = \mathbf{f}^{k-1} + \alpha^k \cdot (\mathbf{fs}^k - \mathbf{f}^{k-1}),$$
$$k = k + 1$$

Figure 3. MSA-FA algorithm steps

$$\mathbf{f}^k = \mathbf{f}(\mathbf{c}^{k-1}),$$
$$\mathbf{cs}^k = \mathbf{c}(\mathbf{f}^k),$$
$$\mathbf{c}^k = \mathbf{c}^{k-1} + \alpha^k \cdot (\mathbf{cs}^k - \mathbf{c}^{k-1}),$$
$$k = k + 1$$

Figure 4. MSA-CA algorithm steps

A convergence index often used for MSA-FA is the average absolute difference over flows: $(\sum_a |f_{S,a}{}^k - f_a{}^{k-1}| / f_a{}^{k-1}) / n$, where n is the number of arcs. A similar index may be defined for MSA-CA. Others indices may be defined based on the maximum difference, possibly excluding arcs with very low flows.

The step size α^k play a key role in the MSA algorithms, influencing their convergence. In this paper, six variations of the MSA_FA are evaluated (see Table 1), varying the nature of step size and the averaging scheme with respect to the arc flows.

- In basic MSA algorithm the step size is $\alpha^k = 1/k$, k being the iterations counter.
- In the Restarting MSA (RMSA, Cantarella et al. 2014) the step size is evaluated as $\alpha^k = 1/k$, but k is not the iteration counter since it is re-initialized after a certain number of iterations. The method used to evaluate k is shown in Figure 5: initially, it is established a value of k_{ini} and a step Δk; while the value k is less than the step, the value of k is increased by one, else the value of the step Δk is increased by one and k is set equal to k_{ini}.

```
kini = 1
k = kini
Δk amplitude
while k  < Δk:
      k +=1
else:
      Δk +=1
      k = kini
```

Figure 5. Evaluating the step size in RMSA algorithm

- In double RMSA (R2MSA) (Di Gangi et al., 2015) the value of the step size is $\alpha^k = 1/k$, and k is re-initialized after a certain number of iterations to a variable value. As shown in Figure 6, it is initially established a value of k_{ini} and amplitude Δk; while the value $(k - k_{ini})$ is less than the amplitude, the value of k is increased by one, else both the value of amplitude Δk and k_{ini} are increased by one and k is set equal to k_{ini}.

```
kini = 1
k = kini
Δk amplitude
while (k - kini) < Δk:
    k +=1
else:
    Δk +=1
    kini +=1
    k = kini
```

Figure 6. Evaluating the step size in R2MSA algorithm

- In Weighted MSA (WMSA) the value of the step size is $\alpha^k = k^{\delta} / \Sigma_{h=1,k} h^{\delta}$, being δ an integer and k the current iteration. As in Liu *et al.* (2007), the values of parameter δ determines the weight assigned to the later iterations: increasing the value of δ, decreases the weight of flow in the iteration.

- In Restart WMSA (RWMSA) the value of the step size is $\alpha^k = k^{\delta} / \Sigma_{h=kini,k} h^{\delta}$, and the value of k is computed as in RMSA (Figure 5).

- In double RWMSA (R2WMSA) the value of the step size is $\alpha^k = k^{\delta} / \Sigma_{h=kini,k} h^{\delta}$, and the value of k is computed as in R2MSA (Figure 6).

Table 1. The values of step size in SUE

Algorithm	α^k
MSA	1/k
RMSA	1/k, with k evaluated as in Figure 5
R2MSA	1/k, with k evaluated as in Figure 6
WMSA	$k^{\delta} / \Sigma_{h=1,k} h^{\delta}$ being d a constant value
RWMSA	$k^{\delta} / \Sigma_{h=1,k} h^{\delta}$, with k evaluated as in Figure 5
R2WMSA	$k^{\delta} / \Sigma_{h=1,k} h^{\delta}$, with k evaluated as in Figure 6

Some studies consider a constant step size (i.e. Powell and Sheffi, 1982; Bar-Gera and Boyce, 2006), others (i.e. Polyak and Juditsky, 1992; Liu *et al.*, 2009) deal with the definition of a variable step size.

Any of the above described MSA-based algorithms may be implemented through flow-averaging (FA) or cost averaging (CA).

4. NUMERICAL RESULTS

In this section some examples of the application to the test system reported in the Fig. 7 of MSA-FA algorithms for solving SUE will be discussed; a comparison with MSA-CA algorithms as well as theoretical considerations on convergence will be addressed in a future paper.

The system in Fig.7 has one origin destination pair, eight transit lines, 7 hyperpaths with one path, and 8 hyperpaths with several overlapping paths. In particular, dotted lines and white nodes represent the underlying pedestrian network, black nodes represent bus stops, continuous line represent on-board arcs, where from the little squared nodes depart/arrive boarding/alighting arcs. The definitions of the lines are depicted in Fig. 8, where the number in square brackets indicates the free-flow travel time of the on-board arc of the line expressed in minutes. The features of each line, in terms of frequency and headway, are summarized in Tab. 2.

Table 2. Features of transit lines

Line	1	2	3	4	5	6	7	8
Frequency [bus/h]	12	6	5	4	4	6	10	360
Headway [s]	300	600	720	900	900	600	5	720

A specific arc cost function is used for each type of arc mentioned above (Fig. 1). Let

t_a (sec) be the free flow time on arc a;
f_a be the flow on arc a;
q_a be the capacity on arc a;
φ_l be the frequency of line l;
$\alpha > 0$, $\beta > 0$, $\gamma \in [0, -2+\sqrt{8} = 0.828]$, $\delta > 1$, $\theta > 0$ be parameters to be calibrated.
(In the following $\alpha = 0.1$, $\beta = 0.2$, $\gamma = 0.7$, $\delta = 2$, $\theta = 0.5$.)

For boarding arcs, cost usually also depends on flow $f_{a'}$ on the corresponding alighting arc a':

$$c_a = t_a [1 + \alpha ((f_a + \gamma (f_{a'} - f_a))/ q_a)^{\delta}]$$

For on-board arcs, cost usually depends on flow to model crowding discomfort:

$$c_a = t_a [1 + \beta (f_a / q_a)^{\delta}]$$

For alighting arcs:

$$c_a = 20 \text{ sec}$$

For waiting arcs, cost depends on the considered hyperpath j, being a specific attribute:

$$c_a^j = \theta / \Sigma_{l \in j} \varphi_l$$

For access/egress or pedestrian arcs as well as connector arcs flow independent cost is assumed given by the travel time with speed equal to 1 m/sec.

All the above arc cost functions respect the existence and uniqueness conditions, if the parameters respect the above conditions.

Total demand is 1000 passengers per hour and the bus capacity (for each line) is 100 (seated or standing) passengers. As already said, hyperpath choice behaviour is modelled through the Probit or the Gammit random utility model. In both cases, the dispersion parameter τ is equal to 0.2.

Each MSA-based algorithm stops either if the convergence index (defined in sub-section 3.2) is less than 0.001 or the number of iterations reaches 1000. Two values of the number of Montecarlo repetitions for computing the arc flow function are considered, 10 and 20.

The bar diagrams in Fig. 9 and Fig. 10 show results obtained for the convergence index, for each combination of hyperpath choice model (Gammit or Probit), type of numbers (Random or Sobol) and MSA algorithms. Tables 3 and 4 show the number of MSA iterations performed.

Table 3. Number of MSA iterations with 10 Montecarlo repetitions

choice model	PRNG	MSA Algorithm					
		MSA	WMSA	RMSA	RWMSA	R2MSA	R2WMSA
GAMMIT	RANDOM	1000	1000	1000	1000	1000	1000
	SOBOL	45	5	12	5	13	5
PROBIT	RANDOM	1000	1000	1000	1000	1000	1000
	SOBOL	45	5	12	5	13	5

Table 4. Number of MSA iterations with 20 Montecarlo repetitions

choice model	PRNG	MSA Algorithm					
		MSA	WMSA	RMSA	RWMSA	R2MSA	R2WMSA
GAMMIT	RANDOM	1000	1000	1000	1000	1000	1000
	SOBOL	44	5	12	5	13	5
PROBIT	RANDOM	1000	1000	1000	1000	1000	1000
	SOBOL	45	5	12	5	13	5

It can easily recognize that all algorithms based on Sobol quasi-random numbers greatly outperform those based on Mersenne Twister pseudo-numbers; moreover all variants of basic MSA allow greatly reducing the number of iterations. It seems that Weighted MSA may outperform Restart MSA, while Double MSA algorithms do not improve over the others. Anyhow further analysis is needed to better compare MSA variants.

5. CONCLUSIONS

In this paper SUE assignment for transit systems, considering pre-trip / en-route path choice behaviour is studied. This problem is relevant for urban transit networks, where travelers may not completely know the status of service, say bus arrivals at stops, and may choose only a travelling strategy at origin. Under mild assumptions such strategies can be modelled as hyperpaths; then, hyperpath choice behavior can be modelled through Random Utility Models applied to the set of (elementary) hyperpaths, such as Probit and Gammit; these models allow for general co-variance matrices but their application requires Montecarlo techniques. Several MSA-based algorithms have been presented and compared on a simple network using two different Pseudo-Random Number Generators: Mersenne Twister and Sobol, showing that all algorithms based on Sobol quasi-random numbers greatly outperform the others, and that MSA variants outperform basic MSA.

In a future paper a comparison of MSA-FA with MSA-CA algorithms as well as theoretical considerations on convergence will be addressed. Other methods to generate pseudo-random numbers needed to Montecarlo technique will also be analyzed. Results of an application to a real-size network will also be discussed.

Also worth of further research effort is the analysis of low frequency transit systems where a user chooses a sequence of runs leading to the so-called scheduled-based assignment; this is often the case for extra-urban trips, on trains, air flights, … .

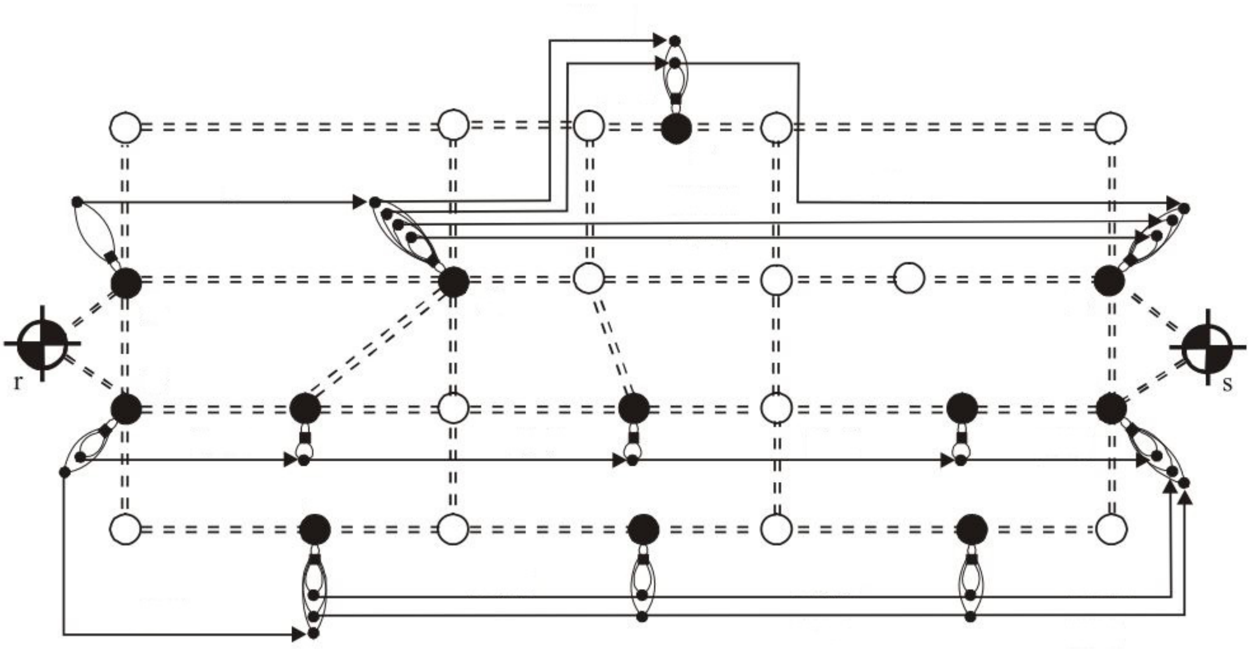


Figure 7. The transit test system

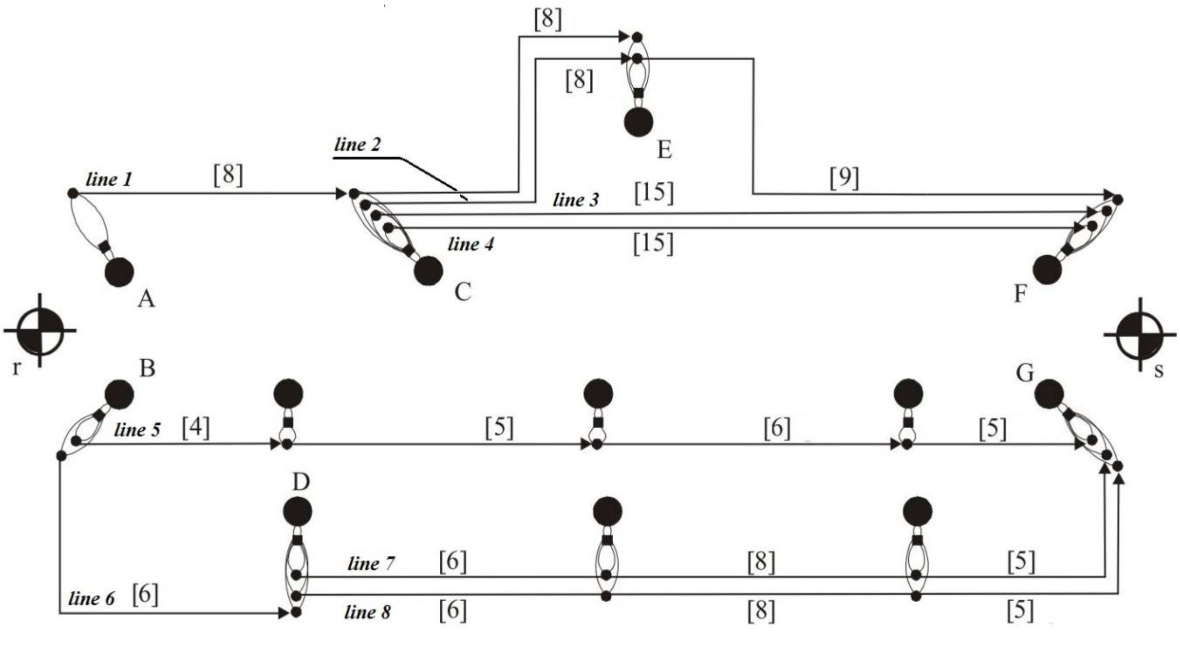
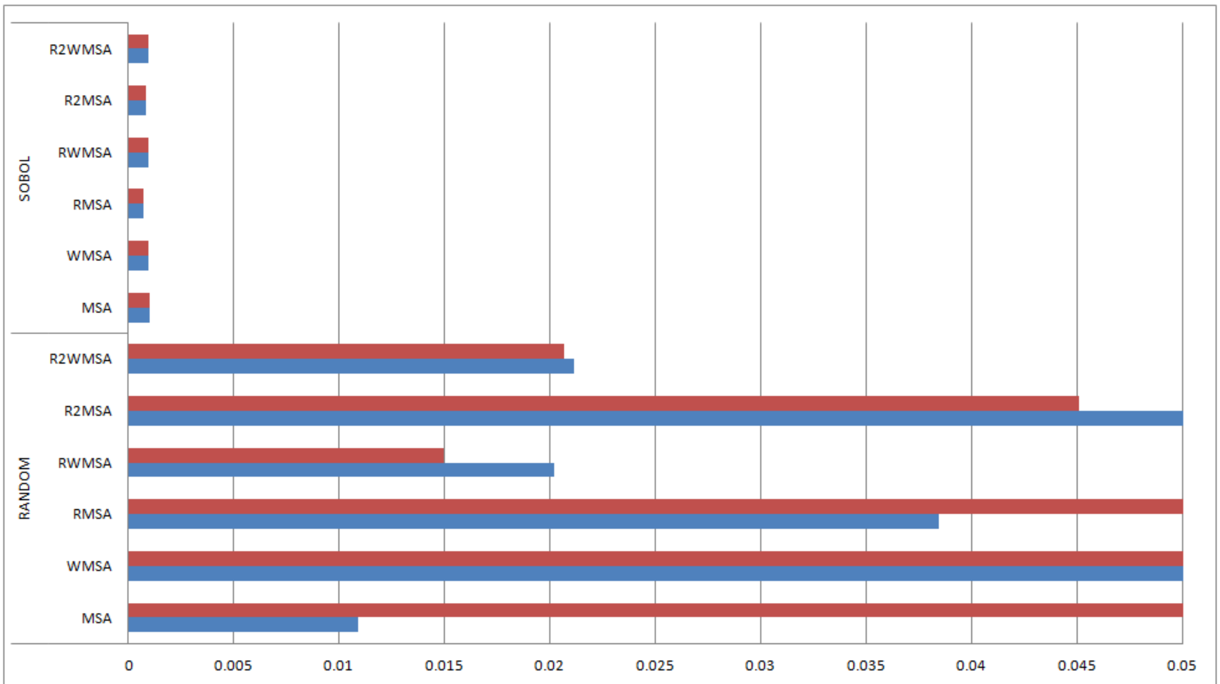
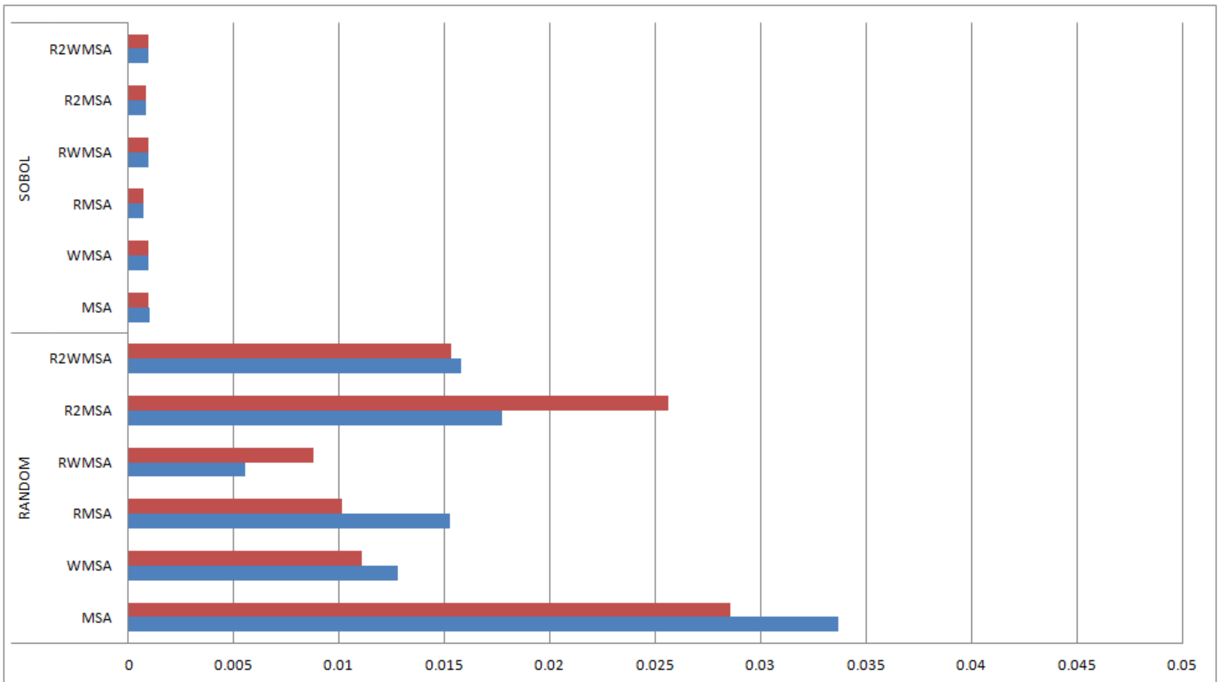


Figure 8. The transit lines



	RANDOM						SOBOL					
	MSA	WMSA	RMSA	RWMSA	R2MSA	R2WMSA	MSA	WMSA	RMSA	RWMSA	R2MSA	R2WMSA
PROBIT	0.114639531	0.081374018	0.056585278	0.014984292	0.045108854	0.020668515	0.000994545	0.0009550120	0.000729493	0.0009550120	0.000833585	0.000955012
GAMMIT	0.010892267	0.138460318	0.038432931	0.020226257	0.058797365	0.021174468	0.000987674	0.0009520350	0.000724455	0.0009520350	0.000827829	0.000952035

Figure 9. Values of the convergence index - internal SNL iteration = 10



	RANDOM						SOBOL					
	MSA	WMSA	RMSA	RWMSA	R2MSA	R2WMSA	MSA	WMSA	RMSA	RWMSA	R2MSA	R2WMSA
PROBIT	0.028559165	0.011065881	0.010136604	0.008784763	0.025614338	0.015298919	0.00098403	0.000950433	0.000721784	0.000950433	0.000824777	0.000950433
GAMMIT	0.033677436	0.01276346	0.015273902	0.005528565	0.017735036	0.015771722	0.000999746	0.000947315	0.000716669	0.000947315	0.000818933	0.000947315

Figure 10. Values of the convergence index - internal SNL iteration = 20

6. REFERENCES

Abramowitz, M., Stegun, I., eds. (1972), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York: Dover Publications, ISBN 978-0-486-61272-0

Arwade S. R., Morad M., Louhghalam A. (2010). Variance decomposition and global sensitivity for structural systems. *Engineering Structures*, 32, pp. 1-10.

Bar-Gera H., Boyce D. (2006). Solving a nonconvex combined travel forecasting model by the method of successive averages with constant step sizes. *Transportation Research Part B*, 40, pp. 51-367.

Bhat C.R. (2003). Simulation estimation of mixed discrete choice models using randomized and scrambled Halton sequences. *Transportation Research Part B*, 37, pp. 837-855.

Bouloré A., Struzik C., Gaudier F. (2012). Uncertainty and sensitivity analysis of the nuclear fuel thermal behavior. *Nuclear Engineering and Design*, 253, pp. 200- 210.

Burrell J.E. (1968). Multiple route assignment and its application to capacity restraint. In Proceedings o the 4t International Symposium on the Theory of Road Traffic Flow, W. Leutzbach and P. Baron eds.. Karlsruhe, Germany.

Cantarella G.E. (1997). A general fixed-point approach to multi-mode multi-user equilibrium assignment with elastic demand. *Transportation Science*, 31, pp. 107-128.

Cantarella G.E. and Binetti M. G. (2002). Stochastic Assignment with Gammit Path Choice Models. In *Transportation Planning* M. Patriksson (ed.), p 53-67. Kluwer Academic Publisher. Printed in the Netherlands.

Cantarella G. E., Vitetta A. (2001). Stochastic assignment to high frequency transit networks: models, algorithms, and applications with different perceived cost distributions. In: *Mathematical Methods On Optimization In Transportation Systems*, Pursula and Niittymaki (Eds.), pp. 109-129.

Cantarella G.E., de Luca S., Di Gangi M., Di Pace R. (2014). Approaches for solving the stochastic equilibrium assignment with variable demand: internal vs. external solution algorithms, *Optimization Methods and Software*, DOI:10.1080/10556788.2014.966823

Daganzo, C. F. (1979). Multinomial Probit: The Theory and Its Application to Demand Forecasting. Academic Press, New York, NY.

Daganzo C. (1983) Stochastic network equilibrium with multiple vehicle types and asymmetric, indefinite arc cost jacobians. *Transportation Science*, 17, pp.282-300.

Daganzo C., Sheffi Y. (1977) On stochastic models of traffic assignment. *Transportation Science,* 11, pp. 253-274.

Di Gangi M., Polimeni A., Cantarella G. E. (2015). Some Considerations on the Convergence of MSA Algorithm in Traffic Assignment Problem. *Working Paper 2015-01 of the Lab of Transportation System Analysis at Dept of Civil Engineering of the University of Salerno*.

Liu, H., He, X., He, B. (2009). Method of successive weighted averages (MSWA) and self-regulated averaging schemes for solving stochastic user equilibrium problem. *Networks and Spatial Economics*, 9(4), pp. 485-503.

Marchi A., Liverani A., Del Giudice A. (2009). Polynomial pseudo-random number generator via cyclic phase. *Mathematics and Computers in Simulation*, 79(11), pp. 3328-3338.

Matsumoto M., Nishimura T. (1998). Mersenne Twister: A 623-dimensionally equidistributed uniform pseudorandom number generator. *ACM Transactions on Modeling and Computer Simulation*, 8(1), pp. 3-30.

Moschopoulos P. G. (1985). The distribution of the sum of independent gamma random variables. *Annals of the Institute of Statistical Mathematics*, 37, 541–544.

Nielsen, O. A. (1997). On the distributions of the stochastic components in SUE traffic assignment models. In *Proceedings of 25th European Transport Forum Annual Meeting,*

Seminar F, pages 77–93.
Nguyen S., Pallottino (1988), "Equilibrium Traffic Assignment for Large Scale Transit Networks", EJOR.
Polyak B.T., Juditsky A.B., Acceleration of stochastic approximation by averaging. *SIAM J. Control Optim.*, 30(4), pp. 838-855.
Powell W.B., Sheffi Y. (1982). The convergence of equilibrium algorithms with predetermined step sizes. *Transportation Science*, 16, pp. 45-55.
Robbins, H., Monro, S. (1951). A stochastic approximation method. *Annals of Mathematical Statistics*, 22(3), pp. 400–407.
Sánchez S., Criado R., Vega C. (2005). A generator of pseudo-random numbers sequences with a very long period. *Mathematical and Computer Modelling*, 42, pp. 809-816.
Sheffi Y. (1985). *Urban transportation networks: equilibrium analysis with mathematical programming methods*. Prentice-Hall, Inc., Englewood Cliffs.
Sobol I. M. (1967). On the distribution of points in a cube and the approximate evaluation of integrals. USSR Computational Mathematics and Mathematical Physics, 7(4), pp. 86-112.
Spiess H. (1984), "Contribution à la théorie et aux outils de planification des réseaux de transport urban", Département d'Informatique et de Recherche Opérationnelle, Université de Montrèal.
Wardrop J.G. (1952). Some Theoretical Aspects of Road Traffic Research. *Proc. Inst. Civ. Eng.,* 2(1), pp.325-378.
Wichmann B.A., Hill I.D. (2006). Generating good pseudo-random numbers. *Computational Statistics & Data Analysis*, 51(3), pp. 1614-1622.
Wu J. H., Florian M. (1993), "A Simplicial Decomposition Method for the Transit Equilibrium Assignment Problem", Annals of Operations Research.
Wu J. H., Florian M., Marcotte P. (1994), "Transit Equilibrium Assignment: A Model and Solution Algorithms", Trans. Science.
Zhang C., Chu J., Fu G. (2013). Sobol's sensitivity analysis for a distributed hydrological model of Yichun River Basin, China. *Journal of Hydrology*, 480, pp. 58-68.