Vickrey's bottleneck model with a cooperative sub-route alternative (extended abstract)

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1. Introduction

The aggregate performances of a road network result from the choices of a great number of independent drivers, all competing to minimize their own individual cost. This competition is extremely inefficient and is responsible for a substantial part of the congestion costs (exactly half of these costs if all users are identical, as shown by Arnott et al. (1990)). However, Vickrey (1969) demonstrated on a simple bottleneck that by imposing the right time-dependent toll, departure times can be shifted in such a way that the cost of congestion are entirely suppressed, without affecting the arrival time at destination. However, large-scale implementations of dynamic pricing schemes are still extremely limited (e.g. Singapore and Stockholm) and the most commonly advanced reason for this lack of success is a poor public acceptability.

This issue, however, can be reasonably put aside if there are free or low cost alternatives, as evidenced by the rapid emergence of HOT (High-Occupancy Toll) lanes in the US. Another way to improve acceptability would be to replace tolling by another selecting criterion, such as vehicle occupancy (see HOV lanes), environment-friendly vehicles, or automated vehicles. By reserving the right amount of road capacity at some desirable time to some pre-defined users, Fosgerau (2011) showed that the cost of the selected users can be considerably reduced without worsening the situation of other users. If we fully extend this reasoning, the travel time wasted in congestion can be entirely suppressed by creating as many time slots as there are users, exactly like with Vickrey's time-dependent toll.

In this paper we address the following problem. The capacity of a freeway (this work can be extended later for urban networks) is divided optimally between two group of users. Selfish users that choose by themselves their departure time from home are obliged to use only some part of the freeway. Cooperative users are willing to collaborate with a central operator towards creating system optimum conditions for the entire freeway. This operator will inform cooperative users when they should depart from home with a guarantee of no congestion cost.

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The main originality of this work resides in our treatment of (un)predictability. In the spirit of the allocation of a part of the capacity to some users by Fosgerau (2011), we consider a scheduling service that allocates a passage time at the bottleneck to its users. The nature of this scheduling service remains voluntarily vague: departure times could be allocated randomly, on a first-registered basis or based on some auction (similarly to railway or airline companies' strategies aimed at filling up their vehicles). However, we acknowledge that such a service might be inconvenient for some people and we introduce an associated cost. This cost, named the "cost of cooperation", can be thought of as the sum of a cost associated to scheduling and of a cost that is specific to the technology used. The cost of cooperation is assumed to be distributed among individuals but to be constant for one individual over time. This new cost then plays the role of the selecting criterion: at equilibrium, users would choose to register to the scheduling service only if by doing so they can reduce their cost associated to travel time and schedule penalty by more than their personal cost of cooperation (such users will be qualified of "cooperative" while the others would be qualified of "independent"). Thus, the service proposed can be considered as some additional alternative between public transit and private vehicles: cooperative users have to comply to the schedule but enjoy their own vehicle, have shorter travel times, and importantly, do not waste time in connecting between different modes of transportation.

While this scheme could already be implemented with traditional vehicles, it might impose itself as the trends toward autonomous vehicles and car-sharing converge. On one hand, the development of autonomous vehicles is likely to lead to the emergence of separated road networks, to avoid interactions with conventional vehicles. On the other hand, car-sharing already imposes some form of cooperation since its users often have to reserve in advance a vehicle for a given time. Thus, these two trends will most likely converge soon, thus creating an ideal framework for the implementation of the scheme described here.

Several versions of this problem are addressed in this paper. In the first part, it is assumed that the demand and capacity splits between the two sub-networks are decision variables and we look for their socially optimal values. These optimal values are shown to exist, to be unique, and numerical values are given for specific distributions of the cost of cooperation. With distributions of the cost of cooperation that were deemed reasonable, it was found that such a scheme could reduce the social cost by approximately 12%.

Then, different user equilibria were considered, depending on the existence of constant tolls and on the objective of the operators setting the tolls. With no toll, the cost of anarchy was found to be relatively small, reducing the gains in terms of social cost by just a few percents. In addition, this loss can easily be avoided by applying a small toll on the independent sub-network. However, if a private operator is given the freedom to set the toll on one of the two sub-networks, these gains in terms of social cost decrease with the capacity the private operator is given to manage and the social cost can become worse than in the original "independent only" situation. A Stackelberg equilibrium was then considered in which the government would first set a toll on the other sub-network, knowing how the private would react. This solution allows the social optimum to be obtained again but at the cost of considerable tolls, which were deemed unacceptable. Finally, such a scheme could be efficient from a social point of view only if both sub-networks are free or managed by the government to maximize social welfare. Only Social Optimum and User equilibrium are addressed here, while the impacts of changes in the capacity of the special lane due to high-performance autonomous vehicles (modeled by the multiplying constant g hereafter) and of profit-maximizing strategies will be reported in the full paper.

2. Social optimum

The social cost can be expressed as a function of the demand and capacity splits by:

$$SC(\hat{\theta}, S_c) = \begin{cases} N\kappa \int_{\underline{\theta}}^{\hat{\theta}} xf(x)dx + \delta \frac{N_c^2}{2gS_c} + \delta \frac{(N-N_c)^2}{S-S_c} & \text{if} \quad (\hat{\theta}, S_c) \in [\underline{\theta}, \bar{\theta}] \times]0, S[\\ N\kappa \int_{\underline{\theta}}^{\bar{\theta}} xf(x)dx + \delta \frac{N^2}{2gS} & \text{if} \quad (\hat{\theta}, S_c) = (\bar{\theta}, S) \\ \delta \frac{N^2}{S} & \text{if} \quad (\hat{\theta}, S_c) = (\underline{\theta}, 0) \end{cases} \tag{1}$$

With $N_c = N_c(\hat{\theta}) = N \int_{\underline{theta}}^{\bar{theta}} f(x)dx$, where N_c and S_c are the cooperative demand and road capacity while N and S are the total demand and capacity. g is a constant such that $g \geq 0.5$, accounting for a different use of the physical space in the case cooperative vehicles are autonomous vehicles. Using the standard notations $\delta = \frac{\beta\gamma}{(\beta+\gamma)}$, where β and γ are the values of earliness and lateness. Finally, f is the probability density function of the normalized cost of cooperation, κ is the related scale and $\hat{\theta}$ is the critical cost of cooperation, such that individuals are cooperative if and only if their cost of cooperation is smaller than $\hat{\theta}$.

First, regardless of the cost of cooperation, one can show the following result.

Proposition 1. *For any given demand split N_c, there exists a unique capacity split that minimizes the social cost. It is given by:*

$$S_c^o = \begin{cases} 0 & \text{if} \quad N_c = 0 \\ \frac{N_c}{\sqrt{2g}(N-N_c)+N_c} S & \text{if} \quad N_c \in]0, N[\\ S & \text{if} \quad N_c = N \end{cases} \tag{2}$$

This capacity split is such that the cooperative sub-network is used $\sqrt{2g}$ times longer than the independent sub-network, it is a continuous function on $[0, N]$ and if $N_c \in]0, N[$, this solution is interior. This result is independent of the cost of congestion.

Then, we only need to assume the following condition to obtain a similar result for the optimal demand split and for the optimal pair of demand and capacity splits.

Condition 1. *The support of the pdf of the cost of cooperation is an interval including* 0*, positive for at least some users, bounded below but not necessarily above.*

By differentiating with respect to the demand split, we obtain:

Proposition 2. *For a given capacity split and for a cost of cooperation verifying Condition 1, there exists a unique demand split that minimizes the social cost. If $S_c \in]0, S[$, then the optimal demand split $\hat{\theta}^o$ is interior and is the unique solution of*

$$\kappa\hat{\theta}^o + \delta \frac{S + (2g-1)S_c}{gS_c(S - S_c)} N_c(\hat{\theta}^o) - \delta \frac{2N}{S - S_c} = 0 \tag{3}$$

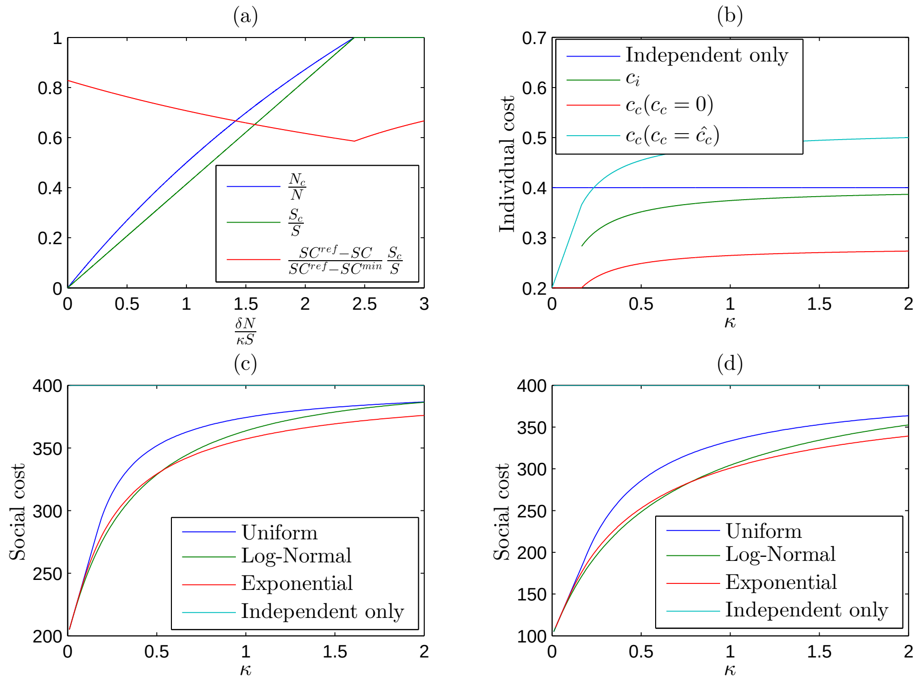

Figure 1: Representations with socially optimal demand and capacity splits with a cost of cooperation that is uniformly distributed on [0, 1] of (a) the demand and capacity splits as functions of the dimensionless ratio $\frac{\delta N}{\kappa S}$, together with the ratio $\frac{SC^{ref}-SC}{SC^{ref}-SC^{min}} \frac{S_c}{S}$ for $g = 1$; (b) individual costs for $g = 1$ for the independent users and for the cooperative users with the smallest and biggest cost, compared with the "independent only" case ; the social cost with different distributions of the cost of cooperation with (c) $g = 1$ and (d) $g = 2.(\frac{N}{S} = 1$ and $\delta = 0.4$ for figures b, c and d)

The implicit function theorem can be used to show that $\hat{\theta}^o$ increases with g and with $\frac{S_c}{S}$, while it increases with $\frac{\delta N}{\kappa S}$ only if $\hat{\theta}^o > 0$ and decreases with $\frac{\delta N}{\kappa S}$ if $\hat{\theta}^o < 0$.

Finally, if both the demand and capacity splits can be varied, the socially optimal pair is found by first expressing the social cost as a function of the demand split only (assuming that the capacity is optimally allocated) and by differentiating this function of one variable.

Proposition 3. *For a cost of cooperation verifying Condition 1, there exists a unique pair of demand and capacity splits that minimizes the social cost. This solution is interior if and only if the maximum value of the cost of cooperation among the population is bigger than* $\frac{\delta N}{S}\frac{\sqrt{2g}-1}{g}$. *In this case, it is given by the implicit equation:*

$$\kappa\hat{\theta} + \frac{\delta}{gS}\left(\left(\sqrt{2g}-1\right)^2 N_c(\hat{\theta}) - \left(2g-\sqrt{2g}\right)N\right) = 0 \tag{4}$$

Else, the socially optimal solution is such that everyone is cooperative.

The implicit function theorem indicates us that if there is an interior solution $\hat{\theta}^o$, it is an increasing function of $\frac{\delta N}{\kappa S}$ and of g, in agreement with our intuition. Explicit expressions can be obtained in Propositions 2 and 3 by assuming a specific distribution for the cost of cooperation. This was done analytically with a uniform distribution and numerically for different distributions all having the same mean cost of cooperation, to provide some idea of the scale of the expected gains. The results are displayed in Fig. 1. The results in Fig. 1b, c, and d were obtained for a peak-hour duration of $1h$ and with all costs normalized with respect to the cost of one time unit of travel time. Even though there is no known value for the cost of cooperation, an educated guess would be that its average is in the order of 0.5 (half an hour of travel time, which is equivalent to $\kappa = 1$). For such a value of κ, a rough estimate for the reduction of the social cost would be approximately 12% (but it could be much more for smaller values of κ). With higher values of g, the social cost can be even further reduced, although this gain is more related to technical progress than to shifts of departure times. The main drawback of such a system is that it makes some cooperative users worse-off, compared to the reference "independent-only" case. We show in the next section that this drawback does not exist under User Equilibrium.

3. User Equilibria

The user equilibrium corresponds to a situation where no user can reduce his individual cost by changing his decision. Thus, if there is no toll:

$$\kappa\hat{\theta}^e + c_c^e = c_i^e \tag{5}$$

where c_c^e is the average individual cost at equilibrium for cooperative users excluding the cost of cooperation and c_i^e is the individual cost at equilibrium for independent users. This case can be simply addressed with the following proposition:

Proposition 4. *For a given capacity split, a demand split satisfies the User Equilibrium condition (Eq. 5) with $\kappa = \kappa^*$ if and only if it is the socially optimal demand split for $\kappa = 2\kappa^*$.*

Thus, all the work that was done for the social optimum remains valid after replacing the value of κ. In particular, there is also existence and uniqueness of a solution for a given capacity split. Another consequence of Proposition 4 is that the user equilibrium can be made socially optimal simply by adding a toll $\tau = \kappa\hat{\theta}^e$ on the cooperative sub-network. Indeed, Eq. 5 becomes $2\kappa\hat{\theta}^e + c_c^e = c_i^e$ and by applying Proposition 4, such a toll leads to the socially optimal split.

While the analysis of the impact of the capacity split on the social cost was relatively easy under the assumption of social optimum, it is much more tedious under User Equilibrium since there is no closed-form relationship between the equilibrium demand and the capacity split (except if the toll is assumed to be socially optimal of course). Thus, the analyses of the price of anarchy, of the equilibrium with a private operator and of Stackelberg equilibria were done by assuming a given distribution (uniform) for the cost of cooperation.

With such an assumption, the price of anarchy was found to be relatively small, as shown in Fig. 2a. In addition, it can be seen in Fig. 2b that user equilibrium makes all users better-off compared with the reference case. These results make the strategy proposed especially well-suited for practical applications, since it does not require any pricing.

In a Stackelberg equilibrium in which the government is leader and sets his toll first, it is always possible for the government to set a toll such that the social cost is minimized, knowing that the private will react by maximizing its profits. However, such an equilibrium leads to extremely high tolls and to exactly the type of situation that was deemed unacceptable by the public and that motivated this work. Thus, the cooperative scheme proposed here has the potential to be acceptable by the public and to reduce the social cost only if there are no tolls or if these tolls obey to a welfare-maximizing strategy rather than to a profit-maximizing strategy.

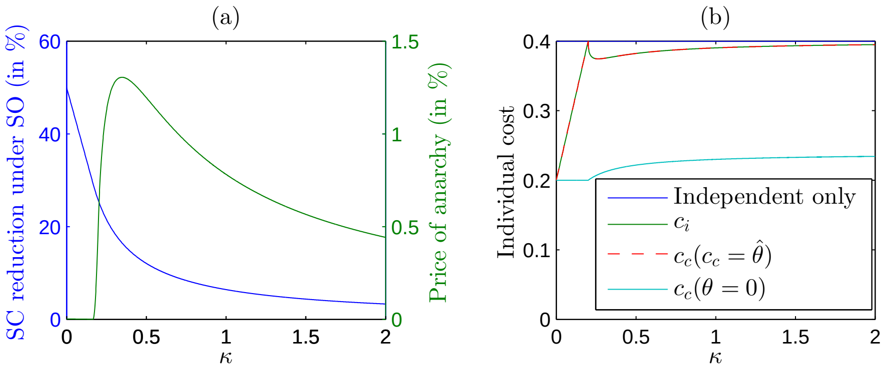

Figure 2: Representation of the price of anarchy under user equilibrium with a socially optimal capacity split (a) and of the associated individual costs (b) for a cost of cooperation that is uniformly distributed on [0, 1], $\frac{N}{S} = 1$ and $\delta = 0.4$.

4. References

Arnott, R., De Palma, A., Lindsey, R., 1990. Economics of a bottleneck. Journal of urban Economics 27, 111–130.

Fosgerau, M., 2011. How a fast lane may replace a congestion toll. Transportation Research Part B: Methodological 45, 845 – 851. doi:http://dx.doi.org/10.1016/j.trb.2011.03.001.

Vickrey, W.S., 1969. Congestion Theory and Transport Investment. American Economic Review 59, 251–60.