Optimal Minimum Delay for Perimeter Traffic Control at an Urban Region

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1 Optimal Control Problem Definition

This paper deals with a perimeter control problem for a homogeneous urban region having a well-defined Macroscopic Fundamental Diagram (MFD), see [1]. The flow dynamic equations for a homogeneous urban region have been already formulated in [3], and they are briefly presented as follows. There are two state variables denoted by $n_{11}(t)$ and $n_{12}(t)$ (veh), which respectively represent the number of vehicles traveling in the region with destination inside and outside the region at time t. The total accumulated number of the vehicles in the region is $n_1(t) = n_{11}(t) + n_{12}(t)$. The MFD links the *accumulation*, $n_1(t)$, and *trip completion flow*, defined as the output flow of the region. The MFD provides low-scatter relationship, if congestion is roughly homogeneous in the region. The MFD is denoted by $G_1\big(n_1(t)\big)$ $(\mathrm{veh/s})$, and it is assumed to be *Lipschitz*, *continuous*, *non-negative*, and *unimodal*. Note that the MFD function should not necessary be a convex function. This assumption is based on many simulation and empirical results, e.g. in [1]. The MFD is defined as the trip completion flow for the region at $n_1(t)$: (i) the sum of a transfer flow, i.e. trips from the region with external destination (outside the region), plus (ii) an internal flow, i.e. trips from the region with internal destination (inside the region). The transfer flow is calculated corresponding to the ratio between accumulations, i.e. $n_{12}(t)/n_1(t) \cdot G_1\big(n_1(t)\big)$, while the internal flow is calculated by $n_{11}(t)/n_1(t) \cdot G_1\big(n_1(t)\big)$.

The traffic flow demands generated in the region with internal and external destinations are respectively denoted by $q_{11}(t)$ and $q_{12}(t)$ $(\mathrm{veh/s})$, while $q_{21}(t)$ $(\mathrm{veh/s})$ denotes a generated traffic flow outside the region with destination to the region.

Following [3], a perimeter control is introduced on the border of the urban region, where its inputs $u(t)$ $(-)$ and $1-u(t)$ control the ratios of flows, $0 \leq u(t) \leq 1$, that cross the border from inside to outside and from outside to inside the region at time t, respectively. Note also that the internal flow cannot be controlled or restricted.

The vehicle-conservation equations in the urban regions are given as follows (same equations (1) and (2) in [3]):

$$\frac{\mathrm{d}n_{11}(t)}{\mathrm{d}t} = q_{11}(t) + (1-u(t)) \cdot q_{21}(t) - \frac{n_{11}(t)}{n_1(t)} \cdot G_1\big(n_1(t)\big)\,, \tag{1}$$

$$\frac{\mathrm{d}n_{12}(t)}{\mathrm{d}t} = q_{12}(t) - \frac{n_{12}(t)}{n_1(t)} \cdot G_1\big(n_1(t)\big) \cdot u(t)\,. \tag{2}$$

Let us now rewrite (1) to have a state equation corresponding to variable $n_1(t)$ instead of $n_{11}(t)$. The reason for that is only technical as this simplifies the mathematical proofs given later. By summing (1) and (2) and substituting $n_{11}(t) = n_1(t) - n_{12}(t)$, one gets the corresponding state equation:

$$\begin{aligned}\frac{\mathrm{d}n_1(t)}{\mathrm{d}t} = q_{11}(t) + q_{12}(t) + q_{21}(t) - \frac{n_1(t) - n_{12}(t)}{n_1(t)} \\ \cdot G_1\big(n_1(t)\big) + \Big(q_{21}(t) + \frac{n_{12}(t)}{n_1(t)} \cdot G_1\big(n_1(t)\big)\Big) \cdot u(t)\end{aligned} \tag{3}$$

Now, we define the optimal control problem for an urban region as follows. The optimal control problem aims at manipulating the control input $u(t)$ to optimize an objective J, subject to (2) and (3). There are a variety of criteria that can be chosen, e.g. the *throughput* of the transportation network and the total network *delay*. In this paper, the network delay is chosen and it is defined as follows:

$$J = \int_{t_0}^{f_\mathrm{f}} n_1(t)\mathrm{d}t\,. \tag{4}$$

where t_0 and f_f [s] are the starting and final times of the control process. Note that the defined optimal control problem here is different than the control problem defined in [3], as in the latter the aim is to achieve maximum throughput by regulating the accumulation $n_1(t)$ around (or nearby) the critical accumulation, where the MFD function has a maximum value.

2 Preliminary results

The optimal solution for the perimeter control problem with minimum total delay is derived utilizing the Krotov-Bellman conditions. The Krotov-Bellman sufficient conditions of optimality are summarized as follows. The reader can refer to [2] for further information. Given a dynamic system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f(t, \mathbf{x}, \mathbf{u})\,, \tag{5}$$

with state variables $\mathbf{x}(t)$, control inputs $\mathbf{u}(t)$, initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, and the following objective function

$$\min J = \int_{t_0}^{f_\mathrm{f}} f_0(t, \mathbf{x}, \mathbf{u})\mathrm{d}t\,, \tag{6}$$

one can construct a function $R(t, \mathbf{x}, \mathbf{u})$ as follows

$$R(t, \mathbf{x}, \mathbf{u}) = \frac{\partial V}{\partial \mathbf{x}} f(t, \mathbf{x}, \mathbf{u}) - f_0(t, \mathbf{x}, \mathbf{u}) + \frac{\partial V}{\partial t} \tag{7}$$

where $V(t, \mathbf{x})$ is assumed to be continuous and differentiable function. Taking into account that the full time derivative of V with respect to (5) is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial \mathbf{x}} f(t, \mathbf{x}, \mathbf{u}) + \frac{\partial V}{\partial t}\,, \tag{8}$$

one can see the identity

$$J = V(f_\mathrm{f}, \mathbf{x}(f_\mathrm{f})) - V(t_0, \mathbf{x}_0) - \int_{t_0}^{f_\mathrm{f}} R(t, \mathbf{x}, \mathbf{u})\mathrm{d}t\,. \tag{9}$$

The sufficient Krotov-Bellman conditions of optimality are as follows: if there exists pair $(\mathbf{x}^*, \mathbf{u}^*)$ such that $\mathbf{x}^*$ is the solution of the dynamic system

$$\frac{\mathrm{d}\mathbf{x}^*}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{x}^*, \mathbf{u}^*)\,, \tag{10}$$

over the time interval $[t_0, f_\mathrm{f}]$, and the following properties

$$\begin{aligned} \mathbf{u}^* &= \mathrm{argsup}_{\mathbf{u}} R(t, \mathbf{x}, \mathbf{u})\,, \\ R(t, \mathbf{x}, \mathbf{u}^*) &= \mu(t)\,, \\ \Theta = V(f_\mathrm{f}, \mathbf{x}(f_\mathrm{f})) &= \text{Constant}\,, \end{aligned} \tag{11}$$

hold, then this pair $(\mathbf{x}^*, \mathbf{u}^*)$ is a global optimal solution. Note that $\mu(t)$ is any measurable bounded function of t. According to these sufficient conditions of optimality the problem is reduced to the solution of the nonlinear Krotov-Bellman PDE for the function $V(t, \mathbf{x})$.

In this paper, we propose the Modified Krotov-Bellman conditions in the following form:

$$\begin{aligned} \mathbf{x}^* &= \mathrm{argsup}_{\mathbf{x}} R(t, \mathbf{x}, \mathbf{u})\,, \\ R(t, \mathbf{x}^*, \mathbf{u}) &= \mu(t)\,, \\ \Theta = V(f_\mathrm{f}, \mathbf{x}(f_\mathrm{f})) &= \text{Constant}\,. \end{aligned} \tag{12}$$

In (12), the maximization of $R(t,\mathbf{x},\mathbf{u})$ over $\mathbf{u}$ is replaced by the maximization of $R(t,\mathbf{x},\mathbf{u})$ over $\mathbf{x}$. Note that in both variants (11) and (12) the resulting function $R(t,\mathbf{x},\mathbf{u})$ after maximization will be a function of time t only.

Taking into account the upper and lower bounds $\overline{n}_1(t)$ and $\underline{n}_1(t)$, obtained from (2)-(3), we respectively get

$$\overline{g}_1(t) = n_1(t) - \overline{n}_1(t) \leq 0\,, \tag{13}$$

$$\underline{g}_1(t) = \underline{n}_1(t) - n_1(t) \leq 0\,. \tag{14}$$

Let $\overline{\lambda}_1$ and $\underline{\lambda}_1$ be the Lagrange multipliers for (13) and (14), respectively. Now, let us choose $V(t, n_1(t), n_2(t)) = C$, where C is a constant, and substituting (4) into (12), one gets

$$sup_{n_1}[-n_1 - \overline{\lambda}_1\overline{g}_1 - \underline{\lambda}_1\underline{g}_1] = \mu(t),\ \Theta = C. \tag{15}$$

From KKT conditions we get

$$-1 - \overline{\lambda}_1\frac{\partial\overline{g}_1}{\partial n_1} - \underline{\lambda}_1\frac{\partial\underline{g}_1}{\partial n_1} = 0 \tag{16}$$

or

$$\underline{\lambda}_1 = 1,\ n_1 = \underline{n}_1. \tag{17}$$

Lemma 2.1 *Consider an ODE system* $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{f}(\mathbf{x},\mathbf{u},t)$*, where* $\mathbf{f}(\mathbf{x},\mathbf{u},t)$ *is Lipschitz and continuous vector-function,* $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$ *are the state variables, and* $\mathbf{u}(t)$ *are measurable bounded control inputs. The upper bound of the solution* $\mathbf{x}(t)$ *with initial conditions* $\mathbf{x}(t_0) = \mathbf{x}_0$ *is denoted as* $\overline{\mathbf{x}}(t)$*. Each component* i *of this bound can be calculated according to the following equation*

$$\frac{\mathrm{d}\overline{x}_i}{\mathrm{d}t} = \sup\nolimits_{\mathbf{u},x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n} f_i(x_1, x_2, ..., x_{i-1},$$
$$\overline{x}_i, x_{i+1}, ..., x_n, \mathbf{u}, t) \tag{18}$$

with the initial condition $\overline{x}_i(t_0) = x_{i,0}$*, where* $f_i(\cdot)$ *is the* i*-component of the vector-function* $\mathbf{f}(\cdot)$*. The lower bound* $\underline{x}_i(t)$ *has to be determined in the same way by just replacing* sup *by* inf *in (18).*

Lemma 2.2 *The lower bounds for state variables* $n_1(t)$*,* $n_{12}(t)$ *are achieved with control* $u(t) = 1$.

Proof The proof is as follows. The infimum over $u(t)$ of the right-hand side of (3) is achieved for $u(t) = 1$. Substitution $u(t) = 1$ into (3), one gets

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = q_{11}(t) + q_{12}(t) - G_1(n_1(t))\,. \tag{19}$$

From Lemma 2.1 it follows that the solution of (19) is a lower bound $n_1(t) \geq \underline{n}_1(t)$.

Hence, from *Lemma 2.2* it follows that the optimal control will be $u = 1$ in the whole interval $[t_0, f_{\mathrm{f}}]$.

Future work

In this abstract, the control inputs $u(t)$ and $1 - u(t)$ are assumed to be coupled (the sum is equal to 1). In the future work, we aim at deriving the optimal policy for a de-coupled control inputs, namely $u_1(t)$ and $u_2(t)$. Moreover, a comparison between the optimal solutions of the two criteria: minimum network delay, which is presented in this paper, and maximum throughput, defined as the total number of vehicles that complete their trips and reach their destination during the time interval (t_0, f_{f}), will be done.

References

[1] Geroliminis, N., Daganzo, C. F., 2008. Existence of urban-scale macroscopic fundamental diagrams: some experimental findings. Transportation Research Part B 42 (9), 759–770.

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