Instability of day-to-day dynamics with positive effects through social interaction

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1 Introduction

Travellers' behaviours in transport systems depend on various types of interactions among users and hence, in addition to understanding individual travel behaviour, it is necessary to analyse these interactions to understand properties of a transport system. There are two types of interactions. Traditionally, transport studies have dealt with interactions called 'congestion', in which users avoid selecting an alternative that is congested because many other users select the same one. On the other hand, recent studies (e.g. Schmöcker et al. 2014) have suggested interactions which have an influence in the opposite direction to congestion, called social interactions (Durlauf, 2001), in which users tend to select an alternative that is popular (i.e. selected by many users). In the following, the former type of (congestion) interactions that tend to disperse users' behaviour are referred to as 'negative interactions', and the latter type of (social) interactions that tend to synchronise users' behaviour are referred to as 'positive interactions'.

The concept of equilibrium has been widely used to analyse transport systems including either one of these types of interactions,. It tries to find a stationary point, called equilibrium, in which every user has no incentive to change their current choice behaviour. Especially for the negative interaction cases (i.e. congested transport systems), properties of equilibrium have been extensively studied.

One of the important properties of equilibrium is stability. Although equilibrium is defined as a stationary point, it may not be preserved against a small perturbation if it is not stable. Proving stability of an equilibrium solution should be necessary to adopt it as a solution representing a situation in the real world.

To analyse stability of equilibrium, one needs to adopt a day-to-day dynamical model. There are various types of day-to-day dynamical models (see e.g. Watling, 1999). They can be classified into two major categories, i.e. discrete time models and continuous time models. Compared to discrete time models, continuous time models are likely to provide stable dynamics as they basically assume smooth changes of the system over days described by an ordinary differential equation (ODE).

Existing studies have suggested that, under the assumption of a continuous-time day-to-day model, a stable equilibrium solution is likely to exist and be attained in systems representing congested road networks (e.g. Smith, 1984). Although several cases with unstable solutions has been found (e.g. Watling, 1996), there exist other stable solutions in these cases. In addition, stability of the equilibrium for supermodular games, which incorporate positive interactions, has been also proven under certain conditions (Hofbauer and Sandholm, 2007). These results imply that the stability is a common property in transport systems with *either* negative or positive interactions.

This study aims to formulate a dynamical model that includes *both* positive and negative effects, and to explore the emergent dynamic properties of this model, especially the potential for stability and instability. The day-to-day dynamics of such models may be especially complicated because there are two forces driving the system in different directions. Analysing such dynamics is important to understand the limitations of the concept of equilibrium and the importance of the dynamical analysis of transport systems. We also aim to consider variety of day-to-day dynamical models to examine its effects on the behaviour of the system. Especially, to incorporate random utility model in the dynamical model, the perturbed best response model (Hofbauer and Sandholm, 2007) is employed, which has been rarely examined in the context of transport systems.

2. Methodology

A transport model consisting of two alternatives and two user groups is constructed. The model describes user' choice behaviour of two transport modes, referred to as mode A and mode B. Users are classified into two groups. They are referred to as 'influential users (or group 1) and 'followers' (or group 2). The numbers of users selecting mode A in groups 1 and 2 at time *t* are denoted by $\mathbf{x} = (x_1(t), x_2(t))$, respectively, and the total number of users in groups 1 and are denoted by $\mathbf{n} = (n_1, n_2)$. Note that this implies that number of users selecting mode B in both groups is $\mathbf{n} - \mathbf{x}$. Utility functions consist of travel cost and social interaction utility. The simplest linear formulation of utility functions is

$$u_A(x_1(t), x_2(t)) = -K(x_1(t) + x_2(t)) + C + J_1 x_1(t) + J_2 x_2(t) \tag{1}$$

$$u_B(x_1(t), x_2(t)) = J_1(n_1 - x_1(t)) + J_2(n_2 - x_2(t)), \tag{2}$$

where $\mathbf{u} = (u_A, u_B)$ are utility functions of modes A and B, respectively, $K > 0$ is a constant representing congestion, $J_1 > J_2 \geq 0$ are constants representing social interaction from users in groups 1 and 2, respectively, and C is a constant.

Day-to-day dynamical models are formulated in the following form as an ODE:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{u}(\mathbf{x}(t))). \tag{3}$$

where $\mathbf{F}$ is a function. The structure of $\mathbf{F}$ varies according to dynamical models, for example, in the perturbed best response dynamics, $\mathbf{F}(\mathbf{x}, \mathbf{u}) = \mathbf{B}(\mathbf{u}(\mathbf{x})) - \mathbf{x}$, where $\mathbf{B}(\mathbf{u})$ returns the numbers of users selecting mode A according to the logistic model with cost $\mathbf{u}$. In addition to the perturbed best response dynamics, Smith dynamics (Smith, 1984) is also adopted with a random utility term incorporated.

Equilibrium solutions are calculated by solving equation $\mathbf{F}(\mathbf{x}, \mathbf{u}(\mathbf{x})) = 0$ and their stability is examined by calculating eigenvalues of Jacobian of $\mathbf{F}$ at the equilibrium point. In addition to stability analysis of the equilibrium points, the basin of attractor for each stable point is examined to understand how the stable solution is robust against external noises.

3. Result and Discussions

The system defined by the linear formulation in Eqns. (1) and (2) has two types of Wardrop equilibrium. One is 'corner solution', in which all users select either mode A or B. The other is 'line solution', in which congestion cost and social-interaction utility are balanced (i.e. utility of two modes are the same). Appearance of both types depends on the settings of utility functions. In the linear formulation case, there are four cases depending on the parameters. Filled circles in Fig. 1 indicate corner solutions and black lines indicate line solutions. Red/blue lines in Fig. 1 indicate directions of the dynamics by Smith dynamics. It can be seen in Case 3 that the border of the basin of attraction (overlapping with the line solution) can be close to the corresponding equilibrium point. This implies that the small perturbation can easily let system go to the other equilibrium point on the other corner, causing unstable dynamics of the system. The implication of this phenomenon is that the small change of behaviour made by influential people can let the behaviour of all users change drastically. Such a phenomenon has not been derived from the models with either one of the positive interaction or negative interaction.

The presentation will also include detailed stability analyses of equilibria with different dynamical models and their basins of attractors. The effect of external fluctuations is also shown.

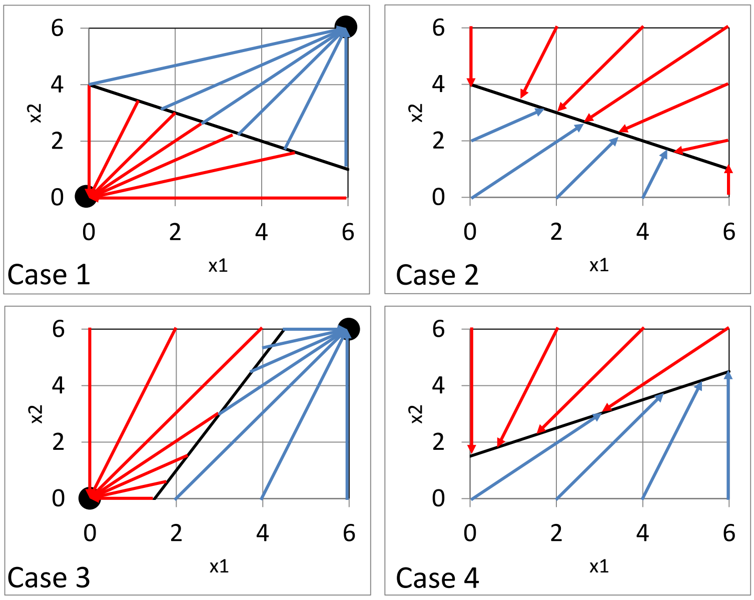

Fig. 1: Equilibrium solutions and directions of dynamics: black circles/lines=equilibria, red/blue lines=direction of dynamics (left-downward/right-upward)

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