Optimal time-dependent patterns of bottleneck permits under stochastic user arrival

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1. Introduction

Most current traffic policies for reducing traffic congestion on urban roads and freeway networks can be divided into two approaches: (a) price-based policies (e.g., congestion pricing; review Yang and Huang, 2005) and (b) quantity-based policies (e.g., trip reservation systems; review Akahane and Kuwahara, 1996; Wong, 1997). Quantity-based policies have a great advantage over price-based policies in the sense that the former directly restrict road usage and so can be implemented without detailed user information. However, there may be cases in which users cannot select their desired choice (e.g., desired arrival time), which would cause economic losses. Therefore, a traffic control policy in which each user can choose his or her desired choice, but does not require detailed demand information, is needed.

As an example of such a policy, Akamatsu *et al*. (2006) proposed a traffic policy named "tradable bottleneck permits (TBP)". This policy consists of two parts: (a) the road manager issues a right that allows the permit holder to pass through a bottleneck during a prespecified time period and (b) the users can select their preferred prespecified time and purchase the corresponding permit in a permit trading market. In this model, the road manager can mitigate queuing congestion by issuing a number of permits for the bottleneck equal to its capacity. Furthermore, Akamatsu *et al*. (2006) proved that the equilibrium under the TBP system achieves a dynamic system optimal (i.e., Pareto optimal) state for a single bottleneck.

Existing research on TBP (e.g., Akamatsu *et al*, 2006; Wada and Akamatsu, 2013) assumes that all users arrive at the bottleneck at exactly the prespecified time. However, in the real world, users arriving early or late relative to the prespecified time could exist. As a result, a queuing delay may occur even if the number of permits issued by the road manager for the bottleneck is equal to its capacity.

This paper examines optimal time-dependent patterns (the number of permits for each time period) for a TBP system with a single bottleneck under the condition that users arrive stochastically. The problem is to find a time-dependent pattern that minimizes the economic losses of all users. To evaluate the economic losses, we employ a disutility model consisting of a queuing delay (i.e., the difference between the users' bottleneck arrival and departure times) and a schedule delay (i.e., the difference between departure and work starting times). Since we assume that the user's arrival is stochastic, the economic losses of all users are evaluated by using their expected disutility.

2. Formulation of Economic Losses

In order to analyze the queuing delay and schedule delay under the condition of stochastic arrival, we assume that the bottleneck arrival rate during each time period follows a normal distribution whose mean is the number of permits for the time period. Under this assumption, we evaluate the time-dependent dynamic queuing length by using the Fokker-Planck equation (see e.g., Newell, 1971).

$$\frac{\partial F(q,t \mid q_0)}{\partial t} = -(\lambda(t) - \mu)\frac{\partial F(q,t \mid q_0)}{\partial q} + \sigma^2 \frac{\partial^2 F(q,t \mid q_0)}{2\partial q^2} \tag{1}$$

where $\lambda(t)$ is the number of permits for the time period t, and μ is capacity of bottleneck. We also assume this utility by quadratic utility function. Then, the expected disutility $E\{U\}$ can be evaluated from the expectation $E\{L_Q\}$ and standard deviation $\sigma\{L_Q\}$ of the queuing delay. Hence, the problem which is to find a time-dependent pattern that minimizes the economic losses of all users is formulated as equation (2):

$$\min_{\lambda(t)} E\{\mathrm{U}\} = \min_{\lambda(t)} \alpha E\{L_Q\} + L_\mathrm{S} + r[\alpha\sigma\{L_Q\}]^2 \tag{2}$$

subject to

$$\int_0^T \lambda(t)dt = N \tag{3}$$

$$\lambda(t) \geq 0 \tag{4}$$

where α and r are the time value and risk aversion, respectively, of all users. Note that here equation (3) is the conservation law for the number of users and that equation (4) is the non-negativity condition for the number of permits for each time period.

3. Determining the Optimal Time-Dependent Pattern by Using Exogenous Issue Pattern

To determine the optimal time-dependent pattern of TBP, we employ two approaches. In the first approach, we consider four types of time-dependent permit issue patterns (the curves shown in Figure 1) in which the shape of each type is fully characterized by a single parameter k, and compare the performances of these patterns by a numerical experiment.

As an example of the experimental results, we show in Figure 2 a contour plot of expected disutility $E\{U\}(k|\sigma)$ under the linear issue pattern with a slope parameter k (i.e., the number of permits per unit time), where σ is the standard deviation of users' arrival rate. From this figure, we can identify an optimal permit issue pattern (i.e., the optimal value of k that minimizes $E\{U\}$) for each (exogenously given) value of σ. In a similar manner, the optimal permit issue patterns for the other three types are completely characterized as functions of σ and r.

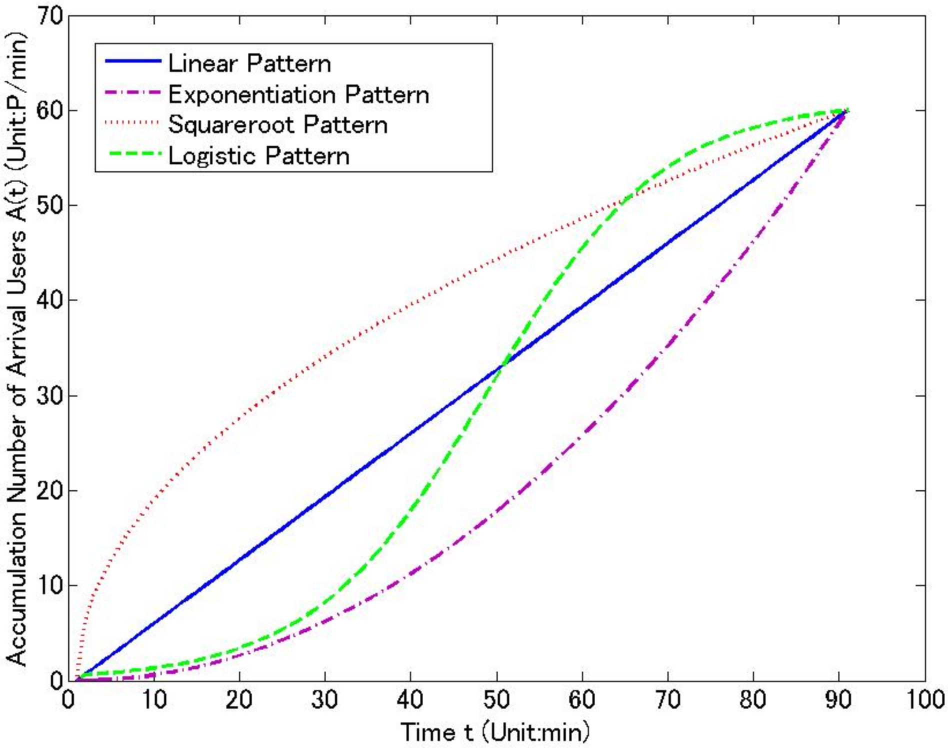

Figure 1: Four types of permit issue patterns

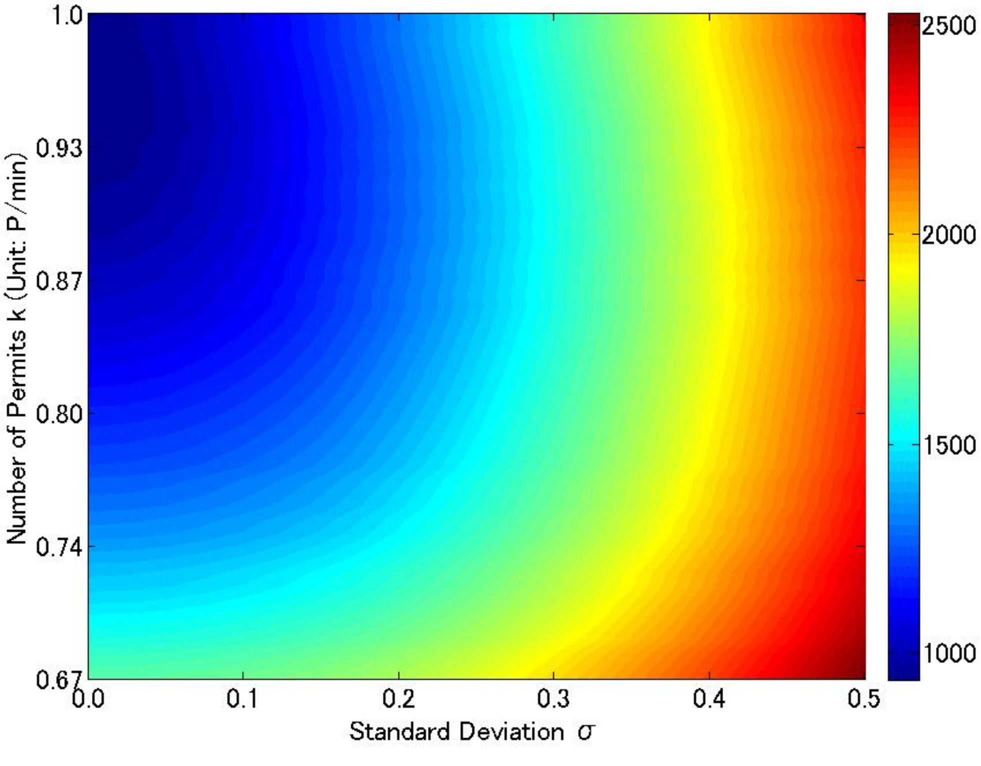

Figure 2: Contour plot of expected disutility (for the case of the linear pattern)

4. Determining the Optimal Time-Dependent Pattern Based on the Stochastic Control Method

To supplement the possible insufficiency of the types of permit issue patterns considered in the first approach, we also show a second approach. In this approach, we formulate the problem as a stochastic optimal control problem in which the number of permits $\lambda(t)$ issued for each time period t is a control variable.

$$\min_{\lambda(t)} \int_0^T \alpha\lambda(t)E\{l_Q(t)\} + \lambda(t)l_S(t) + r[\alpha\lambda(t)\sigma\{l_Q(t)\}]^2 dt \mid q_0 \tag{5}$$

subject to

$$\text{Eq. (3) } \textit{and} \text{ Eq. (4)}$$

$$dq = a(t)q(t)dt + \sigma q(t)dW \tag{6}$$

The objective function (5) is the same as equation (2), but is expressed in terms of the queuing delay $l_Q(t)$ and schedule delay $l_S(t)$ at each time period. Note that the state equation (6) is used to model the stochastic dynamics of $q(t)$ directly. In this equation, the last term $W(t)$ is a standard Wiener process, σ is a volatility parameter, and drift $a(t)$ is governed by equation (7).

$$a(t) = \begin{cases} \{\lambda(t) - \mu\} & if \quad q(t) > 0 \\ \{\lambda(t) - \mu\} & if \quad q(t) = 0 \quad and \quad \lambda(t) \geq q(t) \\ 0 & if \quad q(t) = 0 \quad and \quad \lambda(t) < q(t) \end{cases} \tag{7}$$

Note that equation (7) means that the value of $a(t)$ will change depend on whether a queue exists.

By solving the Hamilton-Jacobi-Bellman equation of the problem, we directly obtain the optimal time-dependent permit pattern $\lambda^*(t)$ (since the details of this approach are somewhat technical, we omit them here). Results of numerical experiments based on this approach demonstrate that the results from the first approach (i.e., under the restriction of the four types of permit issue patterns) are robust.

References

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