INTERNAL VS. EXTERNAL APPROACHES FOR STOCHASTIC EQUILIBRIUM ASSIGNMENT WITH VARIABLE DEMAND

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1. INTRODUCTION

The assignment problem with variable demand has been the subject of several contributions in the past decades. The main contributions may be classified depending on:
i) the approaches to analyze and solve the equilibrium problem.
ii) the choice dimensions considered variable respect to the arc costs (path costs)
iii) the hypothesis on the mutual influence between different transport modes that share the same infrastructure, say multi-mode vs. single-mode assignment.
As regards the approaches, three main modelling options have been pursued: optimization, variational inequalities and fixed-point models, the last resulting the most effective one.
The former approach has been widely used for uncongested network assignment problem and for deterministic user equilibrium by several researcher and is deeply discussed in several text books (Sheffy,1985; Oppenheim, 1995; Bell and Iida, 1997). Optimization models and their extensions (variational inequalities) allow a compact formulation, can rely on several algorithms and can be applied to large scale case study. On the other hand they require simplistic hypothesis on cost functions (separable vs. not separable), on the demand functions, on the route choice models and on the mutual influence between different transport modes.
Fixed-point model approach has been first introduced by Daganzo (1983), who also analyzed variable demand assignment (with the hyper-networks approach) and multi-class assignment. In 1997, Cantarella developed a general treatment of multi-modal/multi-class variable demand equilibrium assignment also for pre-trip/en-route path choice behaviour, including stochastic as well as deterministic user's equilibrium. with fixed-point models and algorithms. Starting from 1997, several methodological contributions have been proposed (such as Bellei et al., 2002; Bar-Gera and Boyce, 2003; D'Acierno et al., 2006). The most uses algorithms are based on Method of Successive Averages (MSA) proposed by Sheffi and Powell (1982) for solving SUE problems. The most followed approach calculates averages on arc flows (MSA-FA) and set the step size according to a predetermined decreasing sequence (1/k, with k being the iteration index). Alternative step size or alternative step techniques have been proposed for solving constant demand assignment problem (Nagurney and Zhang, 1996, Cascetta and Postorino, 2001; Liu et al., 2009). As regards variable demand, most of the existing contributions and/or applications adopt MSA-FA approach with decreasing step or alternative techiniques (Bar-Gera and Boyce, 2006; Liu et al., 2009; Cantarella et al., 2012). Cantarella et al. (2012) propose a systematic comparison between MSA-FA and the MSA approach which calculates averages on arc costs (MSA-CA).
In this paper user equilibrium assignment with variable demand problem is investigated. A fixed-point model approach is pursued and some new algorithms following both the internal and external approaches are proposed and compared with some of those available in literature.

2. PROPOSED METHODOLOGY

As said in the introduction, equilibrium assignment with variable demand may be dealt with through two main approaches: the recently investigated internal approach, which requires to extend existing approaches for assignment with constant demand, and the external approach, which is based on repeated solution of equilibrium assignment with constant demand.

Internal approach algorithms are based on methods of successive averages, and follow either of two approaches: (i) averaging based on *one variable,* (ii) averaging based on *two variables*. The former approach is quite consolidated in literature; the latter approach has never been tested and seems to be more coherent with variable demand assignment problem. As regards averaging with two variables, two options may be pursued, (i) averaging arc flows and path choice satisfaction variables (MSA-FSA) or (ii) averaging demand flows and arc costs (MSA-CDA), that may be proved converging under relatively mild assumptions. Let

$\mathbf{f} \geq 0$ be the arc flow vector;

$\mathbf{c}$ be the arc cost vector, arc cost vector depends on arc flow vector through the cost-flow function $\mathbf{c}(\mathbf{f})$;

$\mathbf{s}$ be the path satisfaction vector (the expectation of the maximum path perceived utility over all the users); path satisfaction depends on path systematic utility values through path satisfaction function that can be expressed as a function of arc costs $\mathbf{s} = \psi(\mathbf{c})$;

$\mathbf{d}$ be the demand flow vector; demand vector depends on path satisfaction vector through the demand flow function $\mathbf{d}(\mathbf{s})$;

S_f be the feasible arc flow set.

Given a feasible set of arc flows $\mathbf{f}^0 \in S_f$ for $k = 0$, and the satisfaction value $\mathbf{s}^0 = \psi(\mathbf{c}(\mathbf{f}^0))$, MSA-FSA may be specified by introducing the following recursive equations [**A**].
Given a feasible set of arc flows $\mathbf{f}^0 \in S_f$ and $\mathbf{c}^0 = \mathbf{c}(\mathbf{f}^0)$ for $k = 0$ with $\mathbf{s}^0 = \psi(\mathbf{c}^0)$ e $\mathbf{d}^0 = (\mathbf{d}(\mathbf{s}^0))$, MSA-CDA may be specified by introducing the following recursive equations [**B**].

[**A**]

$$\begin{cases} k = k + 1 \\ \mathbf{c}^k = \mathbf{c}(\mathbf{f}^{k-1}), \quad \mathbf{d}^k = \mathbf{d}(\mathbf{s}^{k-1}) \\ \mathbf{z}^k = \psi(\mathbf{c}^k), \quad \mathbf{y}^k = \mathbf{f}(\mathrm{c}^k; \mathbf{d}^k) \\ \mathbf{s}^k = \mathbf{s}^{k-1} + (1/\mathrm{k}) \cdot (\mathbf{z}^k - \mathbf{s}^{k-1}) \\ \mathbf{f}^k = \mathbf{f}^{k-1} + (1/\mathrm{k}) \cdot (\mathbf{y}^k - \mathbf{f}^{k-1}) \end{cases}$$

[**B**]

$$\begin{cases} k = k + 1 \\ \mathbf{f}^k = \mathbf{f}(\mathbf{c}^{k-1}; \mathbf{d}^{k-1}), \quad \mathbf{s}^k = \psi(\mathbf{c}^{k-1}) \\ \mathbf{t}^k = \mathbf{d}(\mathbf{s}^k), \quad \mathbf{x}^k = \mathbf{c}(\mathrm{f}^k) \\ \mathbf{d}^k = \mathbf{d}^{k-1} + (1/\mathrm{k}) \cdot (\mathbf{t}^k - \mathbf{d}^{k-1}) \\ \mathbf{c}^k = \mathbf{c}^{k-1} + (1/\mathrm{k}) \cdot (\mathbf{x}^k - \mathbf{c}^{k-1}) \end{cases}$$

The approach pursued is general enough to be extended to every trip choice dimensions, such as trip generation or distribution, and/or mode choice to accommodate most existing demand models and arc cost functions, and it allows easily defining conditions for solution existence and uniqueness, as well as analysis of algorithm convergence.

External approach, algorithms are applied iteratively solving equilibrium between flows and costs for a given demand flow vector, and computing demand flows from demand function for a given cost vector. Let

$\boldsymbol{\varphi}(\mathbf{d})$ be the implicit function between rigid demand equilibrium arc flows and demand flows **d.**

Most existing applications are based on the following recursive equations.

$$
\begin{aligned}
&k = k + 1\\
&\mathbf{f}^{k} = \varphi(\mathbf{d}^{k-1})\\
&\mathbf{c}^{k} = \mathbf{c}(\mathbf{f}^{k})\ ,\ \ \mathbf{s}^{k} = \psi(\mathbf{c}^{k})\\
&\mathbf{d}^{k} = \mathbf{d}(\mathbf{s}^{k})
\end{aligned}
$$

However the above algorithm rarely converges, and is often just stopped few iterations. A more consistent approach is based on MSA, as described below.
Given a feasible set of arc flows $\mathbf{f}^0 \in S_f$ and $\mathbf{c}^0 = \mathbf{c}(\mathbf{f}^0)$ for $k = 0$ with $\mathbf{s}^0 = \psi(\mathbf{c}^0)$ e $\mathbf{d}^0 = (\mathbf{d}(\mathbf{s}^0))$, MSA may be applied to demand flows by introducing the following recursive equations [**C**].
Given a feasible set of arc flows $\mathbf{f}^0 \in S_f$, MSA may be applied to equilibrium arc flows by introducing the following recursive equations [**D**].

[**C**]

$$
\begin{cases}
k = k + 1\\
\mathbf{f}^{k} = \varphi(\mathbf{d}^{k-1})\\
\mathbf{c}^{k} = \mathbf{c}(\mathbf{f}^{k})\ ,\ \ \mathbf{s}^{k} = \psi(\mathbf{c}^{k})\\
\mathbf{d}^{k} = \mathbf{d}^{k-1} + (1/k)\cdot(\mathbf{d}(\mathbf{s}^{k}) - \mathbf{d}^{k-1})
\end{cases}
$$

[**D**]

$$
\begin{cases}
k = k + 1\\
\mathbf{c}^{k} = \mathbf{c}(\mathbf{f}^{k-1}),\qquad \mathbf{s}^{k} = \psi(\mathbf{c}^{k})\\
\mathbf{d}^{k} = \mathbf{d}(\mathbf{s}^{k})\\
\mathbf{f}^{k} = \mathbf{f}^{k-1} + (1/k)\cdot(\varphi(\mathbf{d}^{k}) - \mathbf{f}^{k-1})
\end{cases}
$$

3. NUMERICAL RESULTS

The above described algorithms have been analysed to investigate their performance and to check the effects of different route choice models, such as Multinomial Logit, Mixed Multinomial Logit and Multinomial Probit. Results of the applications to a real-scale urban network (city of Benevento in Italy with a population of 61700, 6400 origin-destination pairs and three considered transportation modes, pedestrian, car and bus) are discussed to support comparison among algorithms and to address main implementation issues.

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