The Hamilton-Jacobi partial differential equation and the three representations of traffic flow

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1. Introduction

Traffic flow on a single link is a surface in the three-dimensional space of vehicle number, time and distance. It can be viewed in three different coordinate systems: Eulerian coordinates (t, x), Lagrangian coordinates (n, x), and (t, n). Each coordinate system gives a different "model" to solve the same problem; these models are called the N-, T- or X-model hereafter, with:

- $N(t, x)$: number of vehicles that have crossed location x by time t,
- $T(n, x)$: time vehicle n crosses x,
- $X(t, n)$: position of vehicle n at time t,

These three models can be expressed as a Hamilton-Jacoby partial differential equation (HJ-PDE) because in all cases there exists a function $\mathcal{H}(\cdot)$–called the Hamiltonian or fundamental diagram in traffic flow–that gives one partial derivatives in terms of the other. The link between conservation laws and HJ-PDE has been known to mathematicians for decades [6, 11, 5], but was brought up to the attention of the traffic flow theory community just recently by [2, 4]. This link is important because it means that the solution of a conservation law (e.g., the kinematic wave model or LWR model of [7], and [12]) can be expressed in terms of solution of the corresponding HJ-PDE formulation, which are much simpler to compute.

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2. Contributions

This paper makes the following contributions:

- Several new and existing models formulated under the same HJ theory.
- Existing models include conservation laws (e.g., cell transmission model [1]), car-following models, cellular automata and mezoscopic models.
- New graphical and exact numerical solution methods for piecewise linear flow-density diagrams, including fixed and moving bottlenecks.
- A new N-model is found that does not require memory (compared to existing formulations).
- A new T-model is found that requires the comparison of only two terms regardless of the shape of the initial and boundary data. This will make stochastic extensions of the kinematic wave model mathematically tractable.
- The addition of a source term and all formulation opens the door for a more rigorous modeling of several practical problems such as merging in Lagrangian coordinates, continuum approximation of merging-divergent traffic in Eulerian coordinates, stochastic extensions and others.

What follows is a general description and main results of the different sections of the proposed paper.

3. General theory

This section presents the HJ theory for a generic "quantity" $U(a,b)$ over generic dimensions a and b. If $G(\cdot)$ gives the value of $U(a,b)$ on a arbitrary curve $\mathcal{B}$ then the scalar and homogeneous HJ-PDE reads:

$$\textbf{HJ}\begin{cases} U_a - \mathcal{H}(U_b) = 0, & \text{(1a)} \\ \qquad U(B) = G(B), \qquad B \in \mathcal{B} & \text{(1b)} \end{cases}$$

were subscript variables represent partial derivatives, B is a point with coordinates (a_B, b_B) in the boundary $\mathcal{B}$, and $G(B) \equiv G(a_B, b_B)$ is the boundary data. The solution of (1) involves the extrema of the function $\phi(\cdot)$:

$$\phi(B, P) = G(B) + (a - a_B)\mathcal{L}\left(\frac{b - b_B}{a - a_B}\right), \tag{2}$$

which is the sum of boundary data and the "cost" on link BP. Here, $P \equiv (a, b)$ is a generic point, $(b - b_B)/(a - a_B)$ is the slope of the line connecting points B and P, and the Lagrangian $\mathcal{L}(\cdot)$ is the Legendre transform of the Hamiltonian. In particular, if $\mathcal{H}(\cdot)$ is concave and $G(\cdot)$ continuous then a solution of (1) is given by:

$$U(P) = \min_{B \in \mathcal{B}^P} \phi(B, P), \qquad \text{(solution for concave } \mathcal{H}) \tag{3a}$$

$$\mathcal{L}(q) = \max_p \{\mathcal{H}(p) - pq\}, \tag{3b}$$

where the auxiliary variable p refers to the argument of the Hamiltonian, i.e. $p \equiv U_b$, and $\mathcal{B}^P$ is the admissible region for B, defined in more detail in the paper.

For convex $\mathcal{H}(\cdot)$ the "max" and the "min" in (3) are interchanged; i.e.:

$$U(P) = \max_{B \in \mathcal{B}^P} \phi(B, P), \qquad \text{(solution for convex } \mathcal{H}) \tag{4a}$$

$$\mathcal{L}(q) = \min_p \{\mathcal{H}(p) - pq\}. \tag{4b}$$

Notice that (4) is new and proved in the paper.

4. Traffic flow models

This section presents the HJ formulation and solution for models $N(t, x)$, $T(n, x)$, and $X(t, n)$ in the case of concave or convex fundamental diagrams. The meaning and symbols used here for the partial derivatives in each coordinate system are summarized in Table 1.

$U(a, b)$	$N(t, x)$		$T(n, x)$		$X(t, n)$	
partials	N_t	$-N_x$	T_n	T_x	X_t	$-X_n$
symbol	$f(t, x)$	$k(t, x)$	$h(n, x)$	$r(n, x)$	$v(t, n)$	$s(t, n)$
name	flow	density	headway	pace	speed	spacing

Table 1: Coordinate systems and variables definition for the three representations

Table 2 summarizes the key definitions for each coordinate system in the context of HJ-PDE theory. $\mathcal{B}^P_{\text{IVP}}$ and $\mathcal{B}^P_{\text{BVP}}$ are the admissible region for IVPs and BVPs, respectively. For simplicity, we shall use the same symbol $\mathcal{H}, \mathcal{L}, \mathcal{B}, p, q, \ldots$ etc for all coordinate systems.

	$N(t,x)$	$T(n,x)$	$X(t,n)$
$\mathcal{H}(p)$	$F(k)$	$H(r)$	$V(s)$
$\mathcal{L}(q)$	$\max_k\{F(k) - k\tilde{v}\}$	$\min_r\{H(r) - r\tilde{s}\}$	$\max_s\{V(s) - s\tilde{f}\}$
$p = \|U_b\|$	k	r	s
$q = \mathcal{H}'$	$\tilde{v}$, wave speed	$\tilde{s}$, wave spacing	$\tilde{f}$, wave flow
HJ-PDE	$f = F(k)$	$h = H(r)$	$v = V(s)$
$\mathcal{H}$ curvature	concave	convex	concave
$\mathcal{B}^P_{\text{IVP}}$	$(x - \hat{q}t, x - \breve{q}t)^+$	$(x - \hat{q}n, x - \breve{q}n)^+$	$(n - \hat{q}t, n - \breve{q}t)^+$
$\mathcal{B}^P_{\text{BVP}}$	$(0, t - x/\breve{q})^+$	$(0, n - x/\breve{q})^+$	$(0, t - n/\breve{q})^+$

Table 2: Key elements of the Hamilton-Jacobi theory for the three coordinate systems. The superscript "+" is introduced to indicate that if a term is negative it should be replaced by 0.

We will show in the paper that the general solution for each of the three models can be expressed as:

$$N(t,x) = \min_{B \in \mathcal{B}^P} \left\{ G(t_B, x_B) + (t - t_B)\mathcal{L}\left(\frac{x - x_B}{t - t_B}\right) \right\} \tag{5a}$$

$$T(n,x) = \max_{B \in \mathcal{B}^P} \left\{ G(n_B, x_B) + (n - n_B)\mathcal{L}\left(-\frac{x - x_B}{n - n_B}\right) \right\} \tag{5b}$$

$$X(t,n) = \min_{B \in \mathcal{B}^P} \left\{ G(t_B, n_B) + (t - t_B)\mathcal{L}\left(\frac{n - n_B}{t - t_B}\right) \right\} \tag{5c}$$

5. Triangular flow-density diagram

In this section we will so that the solution under arbitrary initial and boundary data is given by:

$$N(t,x) = \min\left\{ \min_{y \in \mathcal{B}^P_{\text{IVP}}} \{N(0,y) + Qt - k^*(x-y)\}, N(t - x/u, 0) \right\} \tag{6}$$

$$T(n,x) = \max\left\{T(n,0) + x/u, T(0, x + n\delta) + n\tau\right\}. \tag{7}$$

$$X(t,n) = \min\left\{ \min_{y \in \mathcal{B}^P_{\text{IVP}}} \{X(0,y) + ut - s^*(n-y)\}, X(t - n\tau, 0) - n\delta \right\} \tag{8}$$

where we have defined the free-flow speed u, wave speed $-w$ and jam density κ. Other useful resulting parameters that will be used in the sequel are the capacity $Q = \kappa wu/(w + u)$, the critical density $k^* = Q/u$, the critical spacing $s^* = 1/k^*$, the jam spacing $\delta = 1/\kappa$ and the wave trip time between two consecutive vehicles $\tau = 1/(w\kappa)$.

Findings: 1. even for general initial and boundary data, the solution of the T-model only requires the comparison of two candidates; 2. the BVP solution has only one candidate in all cases; 3. for IVP, the number of candidates for the N- and X-models depends upon the shape of the initial data. A few common cases are discussed in the paper, along with the graphical solution method for all models.

5.1. Exact Discrete models

Here we discretize the previous models in increments $\overset{\leftrightarrow}{n}, \overset{\leftrightarrow}{t}, \overset{\leftrightarrow}{x}$. A "tilde" will denote a dimensionless quantity, e.g. $\widetilde{x} = x/\overset{\leftrightarrow}{x}$ or $\widetilde{X}(\cdot) = X(\cdot)/\overset{\leftrightarrow}{x}$. Two types of discrete implementations are described here: (i) in a "grid" implementation the coordinate system is discrete, e.g. $(\widetilde{t}, \widetilde{x})$, but the dependent variable is continuous, e.g. $X(\widetilde{t}, \widetilde{x})$; (ii) in a cellular automata (CA) implementation everything is discrete, e.g. $\widetilde{X}(\widetilde{t}, \widetilde{x})$.

The grid implementation of the T-model will be shown to be

$$T(\widetilde{n}, \widetilde{x}) = \max\left\{T(\widetilde{n}, \widetilde{x}-1) + \overset{\leftrightarrow}{x}/u, T(\widetilde{n}-1, \widetilde{x}+1) + \overset{\leftrightarrow}{n}\,\tau\right\} \tag{9}$$

and the CA T-model:

$$\widetilde{T}(\widetilde{n}, \widetilde{x}) = \max\left\{\widetilde{T}(\widetilde{n}, \widetilde{x}-1) + 1, \widetilde{T}(\widetilde{n}-1, \widetilde{x}+1) + \theta\right\} \tag{10}$$

with

$$\theta = \frac{u}{w} \quad \text{is an integer.} \tag{11}$$

Similarly, the paper will derive the grid and CA versions of the N- and X-models.

6. Remainder of the paper

Due to space limitations, we simply state the remaining sections:

1. Existing models will be cast as special cases of the models presented here. These include [9, 10, 3, 8] among others to be identified.

2. The conservation law representation of the three models presented so far. The one corresponding to the T-model is new.
3. Addition of a source to the three models. These formulations are new for the T- and X-models.
4. modeling of fixed and moving bottlenecks for all models.
5. modeling of piecewise linear fundamental diagrams for all models.

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