Two-stage column generation and applications

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Abstract

Column generation has been intensively used in the last decades to compute good quality lower bounds for combinatorial problems reformulated through Dantzig-Wolfe decomposition. In this paper we propose a novel framework to cope with problems in which the structure of the original formulation, namely the presence of a combinatorial number of decision variables, does not allow for straightforward reformulation. The basic idea is to start from a meaningful subset of original variables, apply the DW reformulation to the subset, solve the reformulation with column generation and perform the explicit pricing on original variables retracing back the reformulation and using complementary-slackness conditions. The Discrete Split Delivery Vehicle Routing Problem with Time Windows (DSDVRPTW) is used as an illustration for the method, which provides a new exact approach to the problem.

1 Introduction

The resolution of large scale optimization problems arising in real world applications improved in the last decades thanks to the advances in combinatorial optimization theory. Among others, reformulation techniques coupled with Column Generation permit to obtain good quality dual bounds through decomposition (Nemhauser and Wolsey, 1988) and (Desaulniers et al., 2005).

Unfortunately, for a considerable number of optimization problems, their structure does not allow for straightforward reformulation. In particular, the presence in the original formulation of a large, possibly combinatorial, number of decision variables renders the pricing problem of the associated column generation scheme unmanageable. Applications with such characteristics are found in several domains: container terminals (Giallombardo et al., 2008), routing problems (Nakao and Nagamochi, 2007), and scheduling (Xu and Chiu, 2001).

In this paper we present a novel framework called *Two-stage column generation* which can be used for such problems. The basic idea of the framework is simple: we consider a subset of original variables of the so called *compact* formulation, we solve the so called *extensive* formulation via standard column generation and we generate profitable original variables computing their reduced cost, by complementary-slackness conditions, in the same spirit of basic column generation. At the end of this procedure we possibly consider a smaller subset of original variables and identify a sub-set of sub-optimal ones. Since the method addresses linear programs only, i.e. provides dual bounds, we embed it in a branch-and-price scheme as in Barnhart et al. (1998).

The main issue of this approach, addressed in the reminder of the paper, is the computation of the reduced cost of the compact formulation variables, given an optimal solution of the linear relaxation of the extensive formulation. Some attempts in this direction have been proposed with another aim, namely the variable elimination based on reduced cost: for linear integer programs, non-negative variables with a reduced cost greater than the duality gap cannot be positive in any optimal solution, i.e. can be fixed to 0 or eliminated.

Walker (1969) illustrates a method which can be applied if the pricing problem can be solved as a pure linear program. As observed by Irnich et al. (2007), in the case of shortest path subproblems, this results in SPP on acyclic networks. Poggi de Aragão and Uchoa (2003) propose to keep the so-called coupling constraints in the master problem formulation; however, Irnich et al. (2007) observe that there exists a feasible solution for such a master problem where the reduced costs associated to the coupling constraints are all zero, and the authors raise theoretical and algorithmic reasons for not using such approach. Finally, Irnich et al. (2007) propose to estimate the reduced cost of a variable by the smallest reduced cost of a column in which the variable is taken with positive value; this method cannot be directly applied to our two-stage framework since a correct estimation would imply the minimization over the entire set of original variables.

A secondary contribution of our work is a general method to compute reduced costs for original formulation variables when the Dantzig-Wolfe reformulation is applied.

The paper is organized as follows: in section 2, we define the two-stage column generation method. In section 3, we illustrate its application on the Discrete Split Delivery Vehicle Routing Problem with Time Windows and we comment on possible extensions in concluding section 4.

2 General framework

2.1 Dantzig-Wolfe reformulation for integer programs

Consider the following integer linear program, the *original* or *compact formulation* (CF):

$$
\begin{aligned}
z_{IP} = \min \quad & c^T x && (1)\\
\text{s.t.} \quad & Ax \geq b, && (2)\\
& Dx \geq d, && (3)\\
& x \in \mathbb{Z}^n_+. && (4)
\end{aligned}
$$

We assume that conditions to use a standard column generation approach hold, namely the linear relaxation of the Dantzig-Wolfe reformulation of (1)–(4) provides better bounds than the linear relaxation of the original problem, because of the special structure of constraints $\{x \in \mathbb{Z}_+^n : Dx \geq d\}$ which can be easily convexified.

Furthermore, we remark that our framework applies to problems which present a combinatorial number of compact integer variables x. This prevents us to use the standard column generation approach, as the complexity of the resulting subproblem is unmanageable, as we'll see later on.

Let $P = conv\{x \in \mathbb{Z}_+^n : Dx \geq d\} \neq \emptyset$ be a bounded polyhedron. We can represent each $x \in P$ as a convex combination of extreme points $\{p_q\}_{\{q \in Q\}}$ of P:

$$x = \sum_{q \in Q} p_q \lambda_q, \quad \sum_{q \in Q} \lambda_q = 1, \quad \lambda \in \mathbb{R}_+^{|Q|}. \tag{5}$$

The equivalent *extensive formulation* (EF) of (1)–(4) is:

$$\begin{aligned}
z_{IP} = \min \quad & \sum_{q \in Q} c_q \lambda_q & (6)\\
\text{s.t.} \quad & \sum_{q \in Q} A_q \lambda_q \geq b, & (7)\\
& \sum_{q \in Q} \lambda_q = 1, & (8)\\
& \lambda \geq 0, & (9)\\
& x = \sum_{q \in Q} p_q \lambda_q, & (10)\\
& x \in \mathbb{Z}_+^n. & (11)
\end{aligned}$$

where $c_q = c^T p_q$ and $A_q = A p_q \;\; \forall q \in Q$.

If we relax the integrality of x in (11), constraints (10) also become redundant. The resulting *master problem* (MP) is:

$$z_{MP} = \min \quad \sum_{q \in Q} c_q \lambda_q \tag{12}$$

$$\text{s.t.} \quad \sum_{q \in Q} A_q \lambda_q \geq b, \tag{13}$$

$$\sum_{q \in Q} \lambda_q = 1, \tag{14}$$

$$\lambda \geq 0. \tag{15}$$

In column generation we repeatedly solve a *restricted master problem* on a subset of variables λ, which otherwise would be an exponential number. At each iteration we add profitable variables not yet in the formulation, if any, by solving the *pricing subproblem*:

$$\min_{q \in Q} \{\tilde{c}_q := c_q - \pi A_q - \pi_0\} \tag{16}$$

where $\pi \geq 0$ is the dual vector associated to constraints (13), $\pi_0 \in \mathbb{R}$ is the dual variable associated to the convexity constraint (14) and $\tilde{c}_q$ is the reduced cost of variable λ_q.

The resulting pricing is an integer linear program, which eventually exhibits the same computational complexity of the original compact problem. Its complexity is affected by (i) the structure of constraints $\{x \in \mathbb{Z}^n_+ : Dx \geq d\}$, i.e. the nature of extreme points of Q, (ii) the number of decision variables in the compact formulation.

According to our assumptions, variables of the compact formulation are combinatorially many and this results in an unmanageable pricing, which mainly motivates our two-stage column generation approach.

2.2 Two-stage column generation

Let X be the set of compact formulation variables, $|X| = n$. The basic idea of our approach is to start with a subset $\bar{X} \subset X, |\bar{X}| = \bar{n}$ such that (CF) is feasible and iteratively add profitable variables in $\hat{X} := X \setminus \bar{X}$ using reduced costs arguments as in the standard column generation procedure. At each iteration, the resulting master problem is optimally solved using again column generation. The clear benefit of this approach is that the the associated pricing is solved over a smaller set of variables, i.e. the dimension of the vector p_q representing an extreme point is smaller. Furthermore, not all the variables $x_i \in \hat{X}$ will eventually need to be added.

Without loss of generality and for simplicity of notation we assume that $x = [\,\bar{x}\,|\,\hat{x}\,]$, $c = [\,\bar{c}\,|\,\hat{c}\,]$, $A = [\,\bar{A}\,|\,\hat{A}\,]$ and $D = [\,\bar{D}\,|\,\hat{D}\,]$.

The *partial compact formulation* (PCF) is defined as follows:

$$
\begin{aligned}
\bar{z}_{IP} = \min \quad & \bar{c}^T\bar{x} && (17)\\
\text{s.t.} \quad & \bar{A}\bar{x} \geq b, && (18)\\
& \bar{D}\bar{x} \geq d, && (19)\\
& \bar{x} \in \mathbb{Z}_+^{\bar{n}}. && (20)
\end{aligned}
$$

We remark that $\bar{z}_{IP} \geq z_{IP}$.

Let $\bar{P} = conv\{\bar{x} \in \mathbb{Z}_+^{\bar{n}} \,|\, \bar{D}\bar{x} \geq d\} \neq \emptyset$ be bounded. We can represent each $\bar{x} \in \bar{P}$ as a convex combination of extreme points $\{p_q\}_{\{q \in \bar{Q}\}}$ of $\bar{P}$:

$$\bar{x} = \sum_{q \in \bar{Q}} p_q \lambda_q, \;\; \sum_{q \in \bar{Q}} \lambda_q = 1, \;\; \lambda \in \mathbb{R}_+^{|\bar{Q}|} \tag{21}$$

By substituting $\bar{c}_q = \bar{c}^T p_q$ and $\bar{A}_q = \bar{A} p_q \;\forall q \in \bar{Q}$, we can write the equivalent *partial extensive formulation* (PEF), as seen before in (6)–(11); subsequently, by relaxing the integrality constraints on $\bar{x}$, we can define the *partial master problem* (PMP):

$$
\begin{aligned}
\bar{z}_{MP} = \min \quad & \sum_{q \in \bar{Q}} \bar{c}_q \lambda_q && (22)\\
\text{s.t.} \quad & \sum_{q \in \bar{Q}} \bar{A}_q \lambda_q \geq b, && (23)\\
& \sum_{q \in \bar{Q}} \lambda_q = 1, && (24)\\
& \lambda \geq 0. && (25)
\end{aligned}
$$

The resulting pricing subproblem:

$$\min_{q \in \bar{Q}} \{\tilde{c}_q := \bar{c}_q - \pi \bar{A}_q - \pi_0\} \tag{26}$$

is now solvable (due to the lower size of $\bar{Q}$) and column generation can be efficiently applied.

The two-stage column generation approach can be briefly outlined as follows:

Algorithm 1: Two-stage column generation

---

input set $\bar{X}$

repeat

  repeat

    CG1: generate extensive variables λ for partial master problem (PMP)

  until *optimal partial master problem (PMP)* ;

  CG2: generate compact variables x for partial compact formulation (PCF)

until *optimal master problem (MP)* ;

On the one hand, in (CG1) standard column generation applies; in particular, the dual optimal vector π is known at every iteration and thus reduced costs $\tilde{c}_q := \bar{c}_q - \pi\bar{A}_q - \pi_0$ of λ variables can be directly estimated.

On the other hand, in (CG2) we need to know the reduced costs of variables $x_i \in \hat{X}$ in order identify the profitable ones to be added to the partial compact formulation, if any.

Unfortunately, we don't have such information available but we need to reconstruct it from the solution of the partial master problem.

2.2.1 Reduced costs of compact variables by complementary slackness

The dual problem of the linear relaxation of the original compact formulation (1)–(4) is:

$$z_{LR} = \max \quad b^T\alpha + d^T\beta \tag{27}$$

$$\text{s.t.} \quad \alpha A + \beta D \leq c, \tag{28}$$

$$\alpha, \;\; \beta \geq 0. \tag{29}$$

Reduced costs of compact variables x are defined as:

$$\tilde{c} = c - \alpha A - \beta D \tag{30}$$

where α is the dual vector associated to constraints $Ax \geq b$ in (2) and β is the dual vector associated to constraints $Dx \geq d$ in (3).

At each completed cycle of (CG1), we optimally solve the partial master problem and we know the optimal objective function $\bar{z}^*_{MP}$ and the optimal values of primal vector λ^* and dual vector π^*.

We remark that, since π is the dual vector associated to covering constraints (13) in the master problem, it corresponds to dual vector α associated to the same covering constraints (2) in the compact formulation. Similarly for the partial master problem and the partial compact formulation.

Nevertheless, in order to estimate reduced costs (30), we still need to know the dual vector β associated to constraints $Dx \geq d$ of the subproblem in (3).

We propose a method to compute reduced costs of compact variables $x \in \hat{X}$ starting from the optimal solution λ^* of a partial master problem and using complementary slackness arguments.

Firstly, we recall that, given λ^*, we can uniquely reconstruct the equivalent optimal solution $\bar{x}^*$ of the linear relaxation of the partial compact problem. This is trivial when the compact formulation is known and the extensive formulation is obtained through Dantzig-Wolfe reformulation, but this is also possible when only an extensive formulation is given (Villeneuve et al., 2005).

Furthermore, the solution $x = [\,\bar{x}^* \,|\, 0\,]$ is feasible for the linear relaxation of original compact problem (1)–(4), as $\bar{X} \subset X$. Additionally, this solution is trivially optimal for the following constrained problem:

$$\begin{aligned} \min \quad & c^T x && (31) \\ \text{s.t.} \quad & Ax \geq b, && (32) \\ & Dx \geq d, && (33) \\ & \bar{x} = \bar{x}^*, && (34) \\ & \hat{x} = 0, && (35) \\ & x \geq 0. && (36) \end{aligned}$$

The dual of the constrained problem is given by:

$$\begin{aligned} \max \quad & b^T \alpha + d^T \beta + \bar{x}^* \gamma && (37) \\ \text{s.t.} \quad & \alpha A + \beta D + \gamma I + \delta I \leq c, && (38) \\ & \alpha,\ \beta \geq 0, && (39) \\ & \gamma \in \mathbb{R}^{\bar{n}}, && (40) \\ & \delta \in \mathbb{R}^{\hat{n}}. && (41) \end{aligned}$$

Given the optimal solution $x^* = [\,\bar{x}^* \,|\, 0\,]$ of the primal constrained problem, we can compute the associated dual optimal solution $[\,\alpha^*,\ \beta^*, \gamma^*, \delta^*\,]$ using complementary slackness conditions:

$$\begin{aligned} \alpha\,(Ax^* - b) &= 0 && (42) \\ \beta\,(Dx^* - d) &= 0 && (43) \\ \gamma\,(\bar{x}^* - \bar{x}^*) &= 0 && (44) \\ \delta\,(\hat{x}^* - 0) &= 0 && (45) \end{aligned}$$

in addition to dual feasibility:

$$\begin{aligned} \alpha A + \beta D + \gamma I + \delta I &\leq c && (46) \\ \alpha,\ \beta &\geq 0 && (47) \end{aligned}$$

Conditions (44) and (45) just state that variables γ and δ are free, since (34) and (35) are equality constraints. Furthermore, values for vector α are fixed, since we showed that it corresponds to dual vector π obtained solving the (partial) master problem, i.e. $\alpha^* = \pi^*$. Therefore, conditions (42)–(47) reduce to:

$$\beta\,(Dx^* - d) = 0 \tag{48}$$
$$\pi^* A + \beta D + \gamma I + \delta I \leq c \tag{49}$$
$$\beta \geq 0 \tag{50}$$

The dual vector δ represents the reduced costs of not-yet-added variables $\hat{x}$ that we want to determine.

$$[\gamma,\ \delta]^T = \tilde{c} = c - \alpha A - \beta D$$

We observe that (48)–(50) is a system of linear equations which can be transformed in a linear program using a trivial objective function and solved to optimality, e.g. with the simplex method. The analysis and comparison of different objective functions goes beyond the scope of this paper. To illustrate the framework and for the sake of simplicity we propose to use the following objective:

$$\max \mathbf{1^T}\delta$$

Remarks If the solution of (48)–(50) is such that $\gamma^* \geq 0$ and $\delta^* \geq 0$, then $[\,\alpha^*,\ \beta^*\,]$ is also a feasible solution of the dual (27)–(29) of the original compact formulation and therefore $(b^T\alpha^* + d^T\beta^*)$ is a valid lower bound to (1)–(4). Furthermore, since all reduced costs are positive, (CG2) stops.

3 Illustration on the Discrete Split Delivery VRPTW

The Split Delivery Vehicle Routing Problem (SDVRP) is a relaxed version of the classical capacitated VRP in which the number of visits to customer locations is no longer constrained to be at most one, as in the original version of the problem. Thus, in the SDVRP each customer can be visited by more than one vehicle which serves a fraction of its demand. It has been recently shown that this relaxation could yield to substantial savings on the total traveled distance, up to 50% in some instances (Archetti et al., 2006b).

The problem and some properties have been introduced by Dror and Trudeau (1989) with a local search heuristic. Lately Dror et al. (1994) introduce a mathematical formulation based on integer programming and solved through a cutting plane approach. Lower bounds have been proposed by Belenguer et al. (2000) and exact methods have been proposed by Gueguen (1999), Gendreau et al. (2006) and Desaulniers (2008), who address the problem with time windows. Finally, Archetti et al. (2006a) propose a tabu search algorithm.

The Discrete Split Delivery problem is a variant of the SDVRP in which the delivery request of a customer consists of several items which cannot be split further. The problem belongs to the class of split delivery problems since each customer's demand can be split and each customer can be visited by more than one vehicle. This variant arises in some practical applications: Nakao and Nagamochi (2007) present the problem and propose a dynamic programming based heuristic, Xu and Chiu (2001) present the Field Technician Scheduling Problem and propose some heuristics without specifically relate the problem to the DSDVRP, Ceselli et al. (2008) present an exact approach to a real-world VRP in which customers' orders can be split among several vehicles in a discrete fashion. The authors propose a three level order aggregation which end up, at the last level, in considering any possible combination of items.

In the reminder of the section we consider the DSDVRP with time windows (DSDVRPTW) and we assume that, for each customer, all feasible combinations of items are provided as input (which possibly results in a combinatorial number of combinations) and we assume that each vehicle can serve at most one combination per customer. Although very rare, this assumption might lead to a sub-optimal solution because of time windows constraints; the same assumption has been adopted in Ceselli et al. (2008).

Remarkably, service time at customer's location depends on the serviced combination, which is a modeling feature rarely found in literature.

In the next paragraphs we provide some notation and formulations of the DSDVRPTW and apply the two-stage column generation framework to it.

3.1 Compact formulation

We provide a concise and self-explanatory definition of the DSDVRPTW.

Given:

- N : set of customers $\{1, ..., n\}$;
- a_i : minimum arrival time at customer $i \in N$;
- b_i : maximum departure time from customer $i \in N$;
- $G = (V, E)$: complete graph with $V = \{0\} \cup N$;
- c_{ij}, t_{ij} : cost and traveling time of arc $(i, j) \in E$;
- K : set of vehicles of capacity L;
- R_i : set of items to be delivered to customer $i \in N$;
- C_i : set of feasible combinations of items $r \in R_i$ for customer $i \in N$;
- q_i^c : size of combination $c \in C_i$;
- t_i^c : service time of combination $c \in C_i$;
- e_{ic}^r : 1 if item $r \in R_i$ is in combination $c \in C_i$;

we define the following decision variables:

$$x_{ij}^k = \begin{cases} 1 & \text{if arc } (i,j) \in E \text{ is used by vehicle } k \in K; \\ 0 & \text{otherwise.} \end{cases}$$

$$y_{ic}^k = \begin{cases} 1 & \text{if customer } i \in N \text{ is visited by vehicle } k \in K \text{ with combination } c \in C_i; \\ 0 & \text{otherwise.} \end{cases}$$

$$T_i^k = \text{time when vehicle } k \in K \text{ starts service at customer } i \in N.$$

The service time t_i^c for a combination $c \in C_i$ has the following properties:

- $t_i^c \leq \sum_{r \in R_i} e_{ic}^r t^r$
- $t_i^c \geq t^r \;\; \forall r \in R_i : e_{ic}^r = 1$

We propose the following compact formulation for the DSDVRPTW, which aims to minimize the total cost of the arcs traversed by a vehicle and relies on a polynomial number of constraints, a polynomial number of flow (x_{ij}^k) and time (T_i^k) variables and a combinatorial number of selection variables (y_{ic}^k):

$$\min \sum_{k \in K} \sum_{(i,j) \in E} c_{ij} x_{ij}^k \tag{51}$$

$$\sum_{j \in N} x_{0j}^k = 1 \qquad \forall k \in K, \tag{52}$$

$$\sum_{j \in V} x_{ij}^k - \sum_{j \in V} x_{ji}^k = 0 \qquad \forall k \in K, \forall i \in V, \tag{53}$$

$$\sum_{j \in V} x_{ij}^k = \sum_{c \in C_i} y_{ic}^k \qquad \forall k \in K, \forall i \in N, \tag{54}$$

$$\sum_{k \in K} \sum_{c \in C_i} e_{ic}^r y_{ic}^k = 1 \qquad \forall r \in R_i, \forall i \in N, \tag{55}$$

$$\sum_{c \in C_i} y_{ic}^k \leq 1 \qquad \forall k \in K, \forall i \in N, \tag{56}$$

$$T_i^k + \sum_{c \in C_i} t_i^c y_{ic}^k + t_{ij} - T_j^k \leq (1 - x_{ij}^k) M \qquad \forall k \in K, \forall i \in N, \forall j \in V, \tag{57}$$

$$T_i^k - t_{0i} \geq (1 - x_{0i}^k) M \qquad \forall k \in K, \forall i \in N, \tag{58}$$

$$T_i^k \geq a_i \sum_{j \in V} x_{ij}^k \qquad \forall k \in K, \forall i \in N, \tag{59}$$

$$T_i^k + \sum_{c \in C_i} t_i^c y_{ic}^k \leq b_i \sum_{j \in V} x_{ij}^k \qquad \forall k \in K, \forall i \in N, \tag{60}$$

$$\sum_{i \in N} \sum_{c \in C_i} q_i^c y_{ic}^k \leq L \qquad \forall k \in K, \tag{61}$$

$$x_{ij}^k \in \{0, 1\} \qquad \forall k \in K, \forall (i,j) \in E, \tag{62}$$

$$y_{ic}^k \in \{0, 1\} \qquad \forall k \in K, \forall c \in C_i, \forall i \in N, \tag{63}$$

$$T_i^k \geq 0 \qquad \forall k \in K, \forall i \in N. \tag{64}$$

where $M \geq 0$ is a sufficiently large positive real number.

The objective (51) is to minimize the total traveling costs. Flow conservation is ensured by constraints (52)–(54), which also link x and y variables. Customer demands are ensured by covering constraints (55)–(56): remarkably, all items must be covered but not all combinations. Finally, precedence, time windows and capacity constraints are ensured by constraints (57)–(58), (59)–(60) and (61).

We remark the additional complexity incurred by precedence constraints (57) with respect to the same type of constraints in formulations for the VRPTW: the service time at customers location depends on the selection of the combination, i.e. the term $\sum_{c \in C_i} t_i^c y_{ic}^k$ is now a decision variable.

3.1.1 Remarks on notation

The above notation can be simplified by the following remark. By definition no items can be shared among customers i.e. $R_i \cap R_j = \emptyset \ \forall i \neq j$; consequently $C_i \cap C_j = \emptyset \ \forall i \neq j$. We can therefore define the super-sets $R = \bigcup_{i \in N} R_i$ and $C = \bigcup_{i \in N} C_i$, which substitute individual sets $R_1, ..., R_n$ and $C_1, ..., C_n$. Variables y_{ic}^k are substituted by variables y_c^k which are set to 1 if vehicle $k \in K$ delivers combination $c \in C$ to the customer $i \,|\, c \in C_i$ and 0 otherwise. Input data q_i^c, t_i^c and e_{ic}^r are also appropriately redefined as q_c, t_c and e_c^r. Constraints (53)–(60) remain defined for each customer $i \in N$, except the demand satisfaction constraints (55), which can be rewritten as:

$$\sum_{k \in K} \sum_{c \in C} e_c^r y_c^k = 1 \qquad \forall r \in R. \tag{65}$$

3.2 Extensive formulation and standard approach

Consider the simplified notation introduced in 3.1.1. We propose to obtain a lower bound for DSDVRPTW through the Dantzig-Wolfe reformulation of the compact formulation (51)–(64).

For all $k \in K$, let $P^k := conv\{(x_{ij}^k, y_c^k) \,|\, (52) - (54); (56) - (64)\} \neq \emptyset$ be a bounded polyhedron and let $\{p_q\}_{\{q \in Q^k\}}$ be the set of extreme points of P^k, with $p_q = (\bar{x}^q, \bar{y}^q)$. Variables x_{ij}^k and y_c^k can be represented as a convex combination of these extreme points:

$$x_{ij}^k = \sum_{q \in Q^k} \bar{x}_{ij}^q \lambda_q \,, \;\; y_c^k = \sum_{q \in Q^k} \bar{y}_c^q \lambda_q \,, \;\; \sum_{q \in Q^k} \lambda_q = 1 \,, \;\; \lambda \geq 0. \tag{66}$$

We remark that, in this reformulation, each $(\bar{x}^q, \bar{y}^q) \in P^k$ represents a feasible tour for vehicle k delivering a unique combination c to each customer i visited by the the tour.

By definition, vehicles $k \in K$ present identical restrictions (i.e. same capacity): consequently, polyhedra $P^k \equiv P \; \forall k \in K$ and are described by the same set of extreme points $\{p_q\}_{\{q \in Q\}}$. Note that in P the index k disappears from variables x_{ij}^k and y_c^k, as each $(\bar{x}^q, \bar{y}^q) \in P$ represents now a feasible tour that can be covered by whatever vehicle among the $|K|$ available.

After some standard adjustments and aggregation, the linear relaxation of the extensive formulation for DSDVRPTW is the following:

$$\min \sum_{q \in Q} c_q \lambda_q \tag{67}$$

$$\sum_{q \in Q} e_q^r \lambda_q = 1 \quad \forall r \in R, \tag{68}$$

$$\sum_{q \in Q} \lambda_q \leq |K|, \tag{69}$$

$$\lambda \geq 0. \tag{70}$$

where $c_q = \sum_{(i,j) \in E} c_{ij} \bar{x}_{ij}^q$ and $e_q^r = \sum_{c \in C} e_c^r \bar{y}_c^q$. Indices $q \in Q$ correspond to feasible delivery tours, c_q is the cost of tour $q \in Q$ and e_q^r is 1 if item $r \in R$ is delivered in tour $q \in Q$ and 0 otherwise.

Unlike common reformulations for routing problems, partitioning constraints (68) cannot be substituted by convexity constraints unless all possible combinations of items are provided in sets C_i, which is not assumed in this paper.

We remark that, since the polyhedron $P = conv\{(x, y) \,|\, (52)-(54); (56)-(64)\}$ does not possess the integrality property, the relaxation (67)–(70) obtained trough the Dantzig-Wolfe reformulation is stronger than the linear relaxation of (51)–(64).

The master problem is usually solved by means of column generation. However, as we will see, the traditional approach is not applicable in practice, because of the complexity of the resulting pricing subproblem.

The dual formulation of the master problem is:

$$\max \sum_{r\in R} \pi_r + |K|\pi_0 \tag{71}$$

$$\sum_{r\in R} e_r^q \pi_r + \pi_0 \leq c_q \quad \forall q \in Q, \tag{72}$$

$$\pi \in \mathbb{R}^{|R|}, \tag{73}$$

$$\pi_0 \leq 0. \tag{74}$$

The reduced cost of a tour $q \in Q$ is given by:

$$\tilde{c}_q := c_q - \sum_{r\in R} e_r^q \pi_r - \pi_0 \tag{75}$$

The pricing subproblem is then formulated as follows:

$$\min_{q\in Q}\{c_q - \sum_{r\in R} e_r^q \pi_r - \pi_0\} \tag{76}$$

In this pricing problem, two main decisions are made:

1. which customers $i \in N$ are visited by tour $q \in Q$ and in which order: this decision is implicitly represented in the pricing problem through the cost component $c_q = \sum_{(i,j)\in E} c_{ij}\bar{x}_{ij}^q$;
2. which items $r \in R$ are delivered to customer $i \in N$ using which combination $c \in C$: in particular, the decision on combinations is implicitly represented in the pricing problem through the cost component $e_r^q = \sum_{c\in C} e_c^r \bar{y}_c^q$.

Decision (2) is what adds complexity to the subproblem, as combinations $c \in C$ are a combinatorial number.

More specifically, the pricing problem (76) can be formulated as an Elementary Shortest Path Problem with Resource Constraints (ESPPRC) by defining a network $G(\tilde{N}, A)$ which has one node for every customer $i \in N$ and for every combination $c \in C_i$ and whose arcs have transit time equals to $(t_{ij} + t_i^c)$. Since ESPPRC is a NP-Hard combinatorial problem, its solution on such a big network is impractical. A standard column generation approach is therefore not applicable to this problem and, more in general, to the class of problems characterized by a combinatorial number of compact variables which impact on the complexity of the underlying pricing subproblem.

3.3 Application of two-stage column generation to DSDVRPTW

The two-stage column generation framework presented in section 2 can be applied to the extensive formulation for the DSDVRPTW (67)–(70). In particular we consider a subset $\bar{Y} \subset Y$ of y variables and consequently a subset of possible combinations C. We assume w.l.o.g that $Y = [\bar{Y}|\hat{Y}]$ and that $\bar{Q}$ is the set of lower dimension extreme points:

$$\min \sum_{q \in \bar{Q}} c_q \lambda_q \tag{77}$$

$$\sum_{q \in \bar{Q}} e_q^r \lambda_q = 1 \quad \forall r \in R, \tag{78}$$

$$\sum_{q \in \bar{Q}} \lambda_q \leq |K|, \tag{79}$$

$$\lambda \geq 0. \tag{80}$$

We are now interested in computing the reduced cost of a compact formulation variable $y_c^k \in \hat{Y}$ via complementary slackness conditions using the method illustrated in section 2.2.1.

In order to plug DSDVRPTW into the general framework and to comply with the notation used in section 2, we represent all decision variables by a unique vector:

$$x = [\{x_{ij}^k\}_{(i,j) \in E, k \in K} | \{y_c^k\}_{c \in C, k \in K} | \{T_i^k\}_{i \in N, k \in K}]. \tag{81}$$

Matrix A is represented by constraints (65), while the subproblem, represented by matrix D, includes constraints (52)–(54) and (56)–(61).

Dual vector α is associated to matrix A and therefore to constraints (65), while dual vector β associated to matrix D is as follows:

$$\beta = [\,\epsilon \,|\, \zeta \,|\, \eta \,|\, \theta \,|\, \mu \,|\, \tau \,|\, \xi\,] \tag{82}$$

where ϵ is the dual vector associated to constraints (52)–(53), ζ is the dual vector associated to constraints (54), η is the dual vector associated to constraints (56), θ is the dual vector associated to constraints (57), μ is the dual vector associated to constraints (58)–(59), τ is the dual vector associated to constraints (60) and ξ is the dual vector associated to constraints (61).

Using the optimal solution of the PMP, an artificial constrained problem on the basis of (31)–(36), and complementary slackness conditions, we can reconstruct the optimal dual vectors α^* and β^* and subsequently compute the reduced cost of variables y_c^k as follows:

$$\tilde{c}(y_c^k) = 0 + \zeta_i^k - \sum_{r \in R} e_c^r \alpha_r - \eta_i^k - \sum_{j \in V} t_c \theta_{ij}^k - t_c \tau_i^k - q_c \xi^k \tag{83}$$

where the index of the customer i is implicitly represented by the index of the combination c, i.e. $i \,|\, c \in C_i$.

Once we know the reduced cost of variables y_c^k, we can iterate the two-stage column generation framework, with a pricing subproblem that, thanks to the reduction of the dimension, is now solvable.

4 Conclusions

In this paper we present a novel framework called Two-stage column generation to deal with with problems in which the possible combinatorial number of compact formulation variables does not allow for straightforward reformulation.

Starting from a meaningful subset of compact formulation variables and following a standard reformulation scheme we end up with a so called *partial master* formulation and an associated, now manageable, pricing problem. Given an optimal solution of the partial master problem we provide a general method to compute the reduced cost of original variable retracing back the original formulation. Using a modified LP formulation of the original problem and using complementary-slackness conditions we compute the dual variables of the original formulation using a simplex algorithm.

We illustrate the application of the method to the Discrete Split Delivery Vehicle Routing Problem with Time Windows (DSDVRPTW) for which the approach results in the first exact approach known. A comprehensive computational study and analysis on instances of the DSDVRPTW is the subject of an ongoing working paper dedicated to the exact solution of the problem.

The method can be applied to other combinatorial problem and in particular we plan to tackle the Tactical Berth Allocation Problem (TBAP) presented in Giallombardo et al. (2008). In this context, the integration of two planning problems, namely the berth allocation and the quay crane assignment, consists of assigning and scheduling incoming ships to berthing positions, and quay crane to ships. The concept of QC profile discussed in the paper (i.e. number of quay cranes per working shift) defines the service time of a ship. QC profiles are combinatorially many and the selection of a QC profile is a decision variable of the associated compact formulation: the problem seems to be an appropriated candidate for the two-stage framework.

As shown by Irnich et al. (2007) for the VRPTW, a considerable number of arcs (i.e. variables in the original formulation) can be proven to be suboptimal and removed from the formulation. Interestingly, the application of the two-stage column generation framework, i.e. the idea of starting with a meaningful subset of compact formulation variables, might be promising even in this context. We intend to investigate further in this direction.

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