Exploration of the Complex Similarity of Urban System Components

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Abstract

Similar to countless natural phenomena, cities have inherent orders that can be properly captured and expressed through a complex analysis of their components. Using Geographic Information Systems (GIS), this work offers a ring-buffer fractal approach to analyze the spatial characteristics of the components of an urban system. This approach was applied to road length, number of intersections, population+employment, and building gross floor area for the city of Chicago. The complex nature of these four components manifested itself in power-law relationships and represented by their fractal dimensions. Results showed that road length and number of intersections were closely related, albeit their fractal patterns followed slightly different trends. Additionally, population+employment and building gross floor area are significantly similar and one can explain the other. Moreover, the method developed in this study was able to identify the boundary of the old city of Chicago, highlighting its ability to capture hidden characteristics of an urban system. The proposed method could further be used to correlate complex properties of urban transportation systems to other relevant measures, including connectivity, accessibility, and mobility to name a few.

Key words: transportation network, road, intersection, urban system, population, employment, complex analysis, fractal.

1 Introduction

The evolution and spread of an urban system and its components; whether it is its transportation network, or buildings, or even distribution of people themselves, happen over many years. It is the aggregated outcome of numerous individual and collective choices, each influenced by the prevailing conditions in its time. Each new change is overlaid on previous changes. In other words, any urban system and its components have a starting point when and where they were founded; tens or in some cases hundreds of years ago. While it seems reasonable to assume that the older a city is, the less coherent its founding blocks have been, many researchers suggest (Batty and Langley, 1994; Batty, 2008; Batty et al., 2008), and sometimes demonstrate (Chen, 2010a; Doménech, 2009; Friedrich et al., 1994; Rodin and Rodina, 2000; Wong and Fotheringham, 1990), that no matter how an urban system has evolved or what foundations it was built on, from a larger perspective it has inherent order and organization. Having said that, and to better understand the complex nature of an urban system, studies have been focused on characterization of its components rather than itself as a whole (Hillier and Hanson, 1984), because "understanding the topology of urban networks that connect people and places leads to insights into how cities are organized" (Samaniego and Moses, 2008).

The hidden, and presumably orderly, characteristics of different components of a given urban system have been a matter of interest in recent time (Levinson, 2007; Shen, 2002; Terzi and Kaya, 2011). In the case of the transportation network of a city, one can visually observe that such an order manifests itself in a self-similar pattern (Benguigui, 1995; Kim et al., 2003). In other words, a transportation network is very similar to a tree stem that grows, then splits into branches, and then each branch grows and then that also splits into more sub-branches, and so on so forth. One main difference, though, is that transportation networks create loops through branch-joining. Additionally, and noteworthy, order can manifest itself by showing similar shapes and patterns even if scales differ. This is particularly true in road networks that tend to be denser in downtowns, while keeping the same overall pattern of intersections throughout a city.

Nonetheless, the self-similar characteristic of urban systems is not restricted to their transportation networks. In fact, the spread of other components of an urban system, such as population (Appleby, 1996; Chen, 2010b; Lu and Tang, 2004), employment, and buildings (Batty et al., 2008; Frankhauser, 1998a), can show self-repeating patterns as well. This phenomenon clearly fits within the realm of a branch of complex system analysis, i.e. fractals. Indeed, as spatial and non-isomorphic systems with self-repeating patterns, cities clearly exhibit the presence of fractal entities. While this complex behavior of cities and their components has previously been studied (Batty and Langley, 1994; Samaniego and Moses, 2008), new technologies and more disaggregate datasets (and in particular extensive Geographic Information Systems (GIS) data) allow for a more detailed and comprehensive inspection.

Based on the above discussion, the objectives of this work are to: (i) analyze the characteristics of different components of a given city using a proposed fractal approach, (ii) determine the similarities and differences between the complex representations of those components, and (iii) explore and explain the reasons behind such similarities and differences.

Overall, this work fits within the global endeavor to analyze cities and their infrastructure as complex systems (Doménech, 2009; Batty, 2005; Bettencourt et al., 2010; Bettencourt et al., 2007; Derrible, 2012; Kennedy, 2011; Levinson, 2012; Derrible and Kennedy, 2009, 2010). Taking a fractal approach to analyze an urban system and its components offers many benefits, including the provision of a measurable metric. As we will see in the next section, each fractal possesses a particular dimension. As a result, although different systems are not directly comparable (e.g., people versus buildings), comparing their fractal dimensions offers a pragmatic means to gage how they coexist and interact within the built environment.

2 Methodology

2.1 Definition

A fractal can be described as an entity that possesses self-similarity on all scales. It is important to note that a fractal needs to only exhibit similar type of, but not exactly the same, structure at all scales. Moreover, according to Mandelbrot (Mandelbrot, 2004): “A fractal set is one for which the fractal dimension strictly exceeds the topological dimension”. In practice, this means that while a line feature (e.g. a road) has a dimension of 1 in classical geometry, it must have a dimension larger than 1 if it is to have fractal properties.

The rough description of fractal dimension (as used in this work) of a fractal object is the exponent in the expression of the form shown in Eq. 1 (Frankhauser, 1998b; Rasband, 1990):

$$N(r) = ar^{-D} \tag{1}$$

in which r is the radius (with respect to a point of origin or center), N is the number quantifying the object x under consideration at the radius r, a is a constant, and D is the fractal dimension.

2.2 Box-Counting Method

One of the approaches conventionally used in analyzing the properties of a fractal object is the box-counting method (Shen, 2002; Lu and Tang, 2004; Song et al., 2007), as shown in Figure 1.

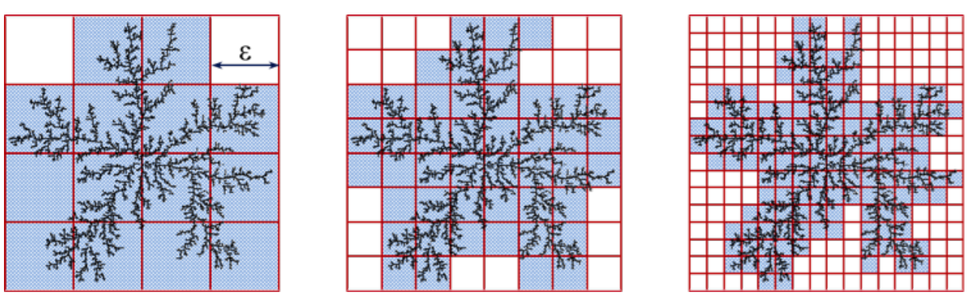


Figure 1: Box-Counting Method, adapted from Biehl (2008)

The idea is that for a self-similar system one should be able to find smaller parts, i.e. "boxes", which demonstrate its repeating pattern. Knowing the geometry of the "boxes" will then result in the determination of the fractal scale. Since the fractal scale is not known at the beginning, however, the size of such "boxes" cannot be readily determined. Thus, the box-counting method turns into an attempt to optimize the way the system is broken into smaller parts, which will eventually result in the determination of the fractal dimension. While this method has been used extensively in various fields (Song et al., 2007; Lovejoy et al., 1987; Liebovitch and Toth, 1989; Sarkar and Chaudhuri, 1994; Smith et al., 1996; Tanaka et al., 1999; Foroutan-pour et al., 1999; Liu et al., 2003; Labedz and Ozimek, 2011), a study by us (to be published soon) has shown that box-counting method is incapable of properly capturing and extracting the fractal nature of physical systems such as transportation networks. For this reason, the following alternative method is developed, applied, and presented here.

2.3 Proposed Ring-Buffer Method

The proposed ring-buffer method is based on the assumption that urban systems evolve similar to living organisms. A living being comes to life as a single cell; let's call it the "center". Then it grows and spreads around that center, subject to its prevailing conditions and constraints. Similar to that, a city spreads around a point of origin, or "center" (Frankhauser, 1998b; Levinson and Xie, 2011), and then gradually expands outwards, while avoiding the physical constraints around it such as water bodies, valleys, etc. The assumption is that the spread of any component of the system, e.g. its road network, at a given point is proportional to its distance from that center. Having determined its "center", the urban system can then be split into rings or buffers around it, as shown in Figure 2. The "center" could be a point, or a small area, around which the urban system has grown and evolved. By calculating the quantity of any given component of the system as a function of the distance from the center, one should be able to verify its fractal nature (if any) and extract its fractal dimension (Tang, 2003).

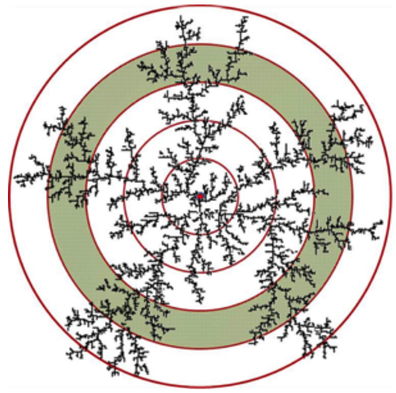


Figure 2: Rings creation in the proposed method

To demonstrate a fractal property, a power law relationship must be present between the quantity of the component under consideration and the radius, as discussed in Eq. 1. Nevertheless, because certain areas have to be excluded (i.e. lakes, rivers, airports, etc.), the ring areas differ from each other significantly. As a result, and instead of the quantity of the component within each ring, its cumulative density within the entire circular buffer is calculated (i.e. eq.1 is integrated and then divided by the area). Consequently, and as shown in Eq. 2, the density of a fractal entity within a circular buffer area also follows power law of the form:

$$\text{Density}\,(r) = br^{-(D-1)} \tag{2}$$

where b is a constant.

Showing a power law trend by the density of the fractal object will enable one to extract the fractal dimension of the object from the slope of the log-log plot of the data as per Eq. 3:

$$\ln(\text{Density}\,(r)) = \ln(b) - (D-1)\ln(r) \tag{3}$$

i.e. $D = \text{slope} + 1$. The regression technic to be used for Eq. 3 is sufficient in this case since there are only few data points; the reader is referred to Clauset et al. (2009) for a further discussion regarding statistical methods that can be used to fit power laws.

Finally, using this proposed ring-buffer approach, a question naturally arises about the selection of the center of a city, especially in cases of mono- versus poly-centric urban forms. This is an ongoing debate that does not have a definite answer to date. It is, however, irrelevant in our case since the methodology is applied to Chicago that has a well-defined center, as it will be seen in the next section.

3 Application

3.1 Case study: Chicago

This work attempts to investigate whether the spread and evolution of one of the oldest cities in North America, i.e. Chicago, has an inherent fractal nature. Having verified the hypothesis, Eq. 3 will then be used to find the fractal dimensions of its components. What made Chicago a unique choice was not only its long history during which it had experienced different periods of urban evolution, but also its unique topology. Chicago is restricted on the east side by Lake Michigan (Figure 3), which means it has only been able to expand towards the west. Moreover, two branches of Chicago River run through it from north and south which join together to the west of the center of the city and then run eastward towards Lake Michigan. Such natural constraints on the evolution of its urban system, in addition to the man-made barriers such as its two international airports, offer intriguing challenges to the process of applying the proposed model.

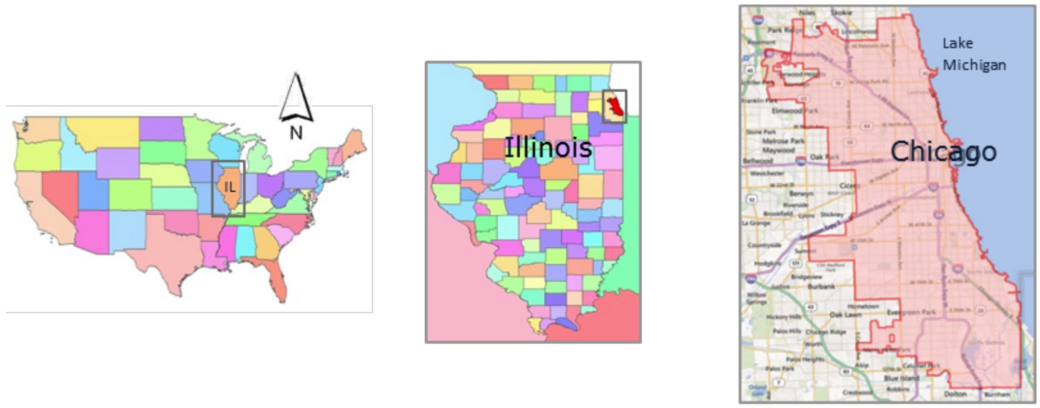


Figure 3: Location Map for Chicago, Illinois (Background: Bing Maps Hybrid)

Moreover, Chicago has a well-defined center, called the “Loop”. The “Loop” is Chicago’s Central Business District (CBD), hosting the Chicago Mercantile Exchange, as well as being the city’s administrative center. It is a well-defined 1.0 km x 1.2 km rectangle surrounded by freeways and partially the Chicago River (Figure 4).

Figure 4: The “Loop”, CBD of Chicago (Background: Bing Maps Hybrid)

For that reason, the “Loop” seemed to be a natural choice for the “center” (or the “heart”) of the city. Having done so, rings were then created around it with radii from 1 km to 21 km, in increments of 1 km (Figure 5). As shown, the physical or natural barriers to the city’s expansion, such as the lake, rivers, valleys, airports, etc., have been cut out of the rings.

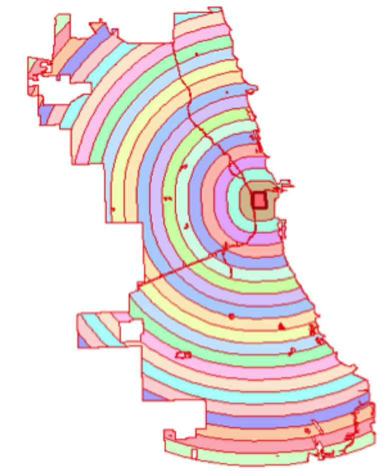


Figure 5: Rings, splitting Chicago into rings from the center

As for the urban system component to be studied, the road network is a natural choice. That being said, several attributes of roads can be studied. The first attribute to consider is the road length, due to the fact that roads are visually similar to fractals. As a second attribute, the intersections within the road network are hypothesized to have fractal nature, as they are directly related to the roads.

A less obvious choice was the building area within the city. The rationale was that the construction of buildings and their spread resembles the expansion of living organisms. They start from a central area, and then spread and expand in different directions while avoiding natural, as well as man-made, barriers. Moreover, since a building not only spreads horizontally, but also vertically, and because its footprint alone could not capture that characteristic, total building gross floor area (equal to the footprint multiplied by the corresponding number of floors) was chosen as another urban system component to study.

Finally, population+employment attributes were selected as the last candidate for fractal analysis. This essentially means that population and employment numbers are summed together because the downtown tends to host few households but many jobs, while the opposite is true as we move towards suburbs. The rationale for including population+employment is that the buildings (i.e. a supply) are constructed to meet a demand, and this demand is mostly (though not completely) generated by the needs to live and work. Because of that, similar properties between roads, intersections, building gross floor areas, and population+employment are expected. This is particularly true for buildings and population+employment, where we expect a close relationship.

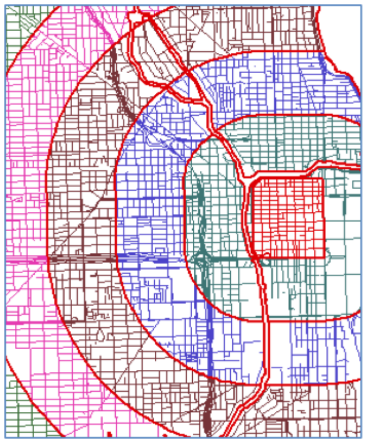
3.2 Data

The data for this study was obtained from different sources, as shown in Table 1.

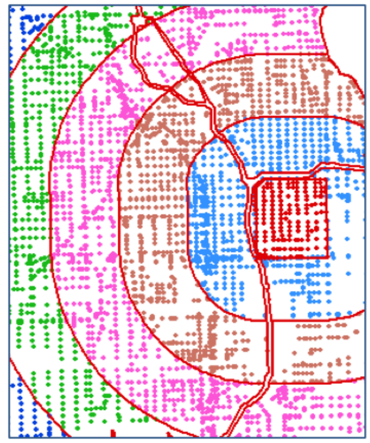
Data	Source
Road Network	U.S. Census Bureau, TIGER/files
Census Tracts and Population Data	U.S. Census Bureau, American FactFinder
Building footprints, Land-Use, and Employment Data	Chicago Metropolitan Agency for Planning

Table 1: Data sources

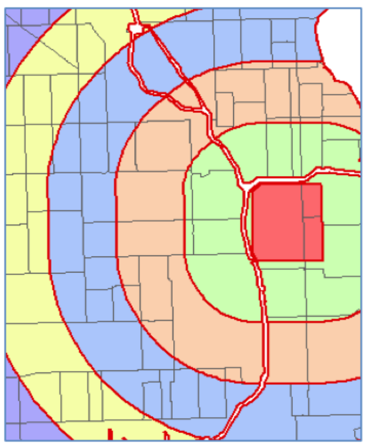
Having obtained the data for each of the four urban system components, their quantities in each ring and the areas of the rings are calculated. That information is then used to calculate the densities of those components for buffers around the center at the selected radii (Figure 6).



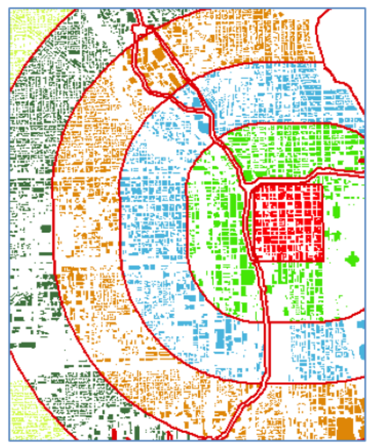
6a: Road length



6b: Intersections



6c: Population + Employment



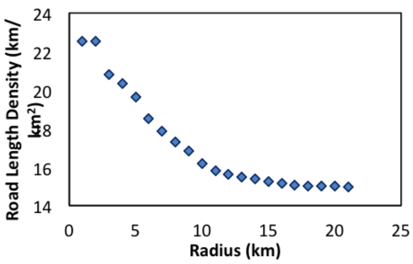
6d: Gross Floor Area

Figure 6: Chosen urban components within equi-distance rings around the Center

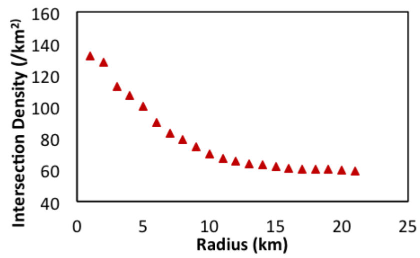
4 Results and analysis

4.1 Results

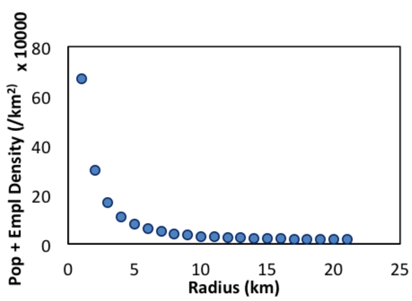
In order to come up with the metrics that are comparable at different radii, the density (Eq. 2, as opposed to the count N from eq. 1) of each component was calculated for buffers at the radii from 1 km to 21 km at increments of 1 km. The results are plotted in Figure 7. The diagrams provide an opportunity to observe the variations in the overall cumulative (average) value of any given component as a function of the distance from the center.



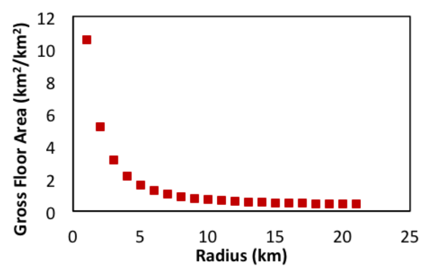
7a: Road length Density vs. Radius



7b: Intersections Density vs. Radius



7c: Population + Employment Density vs. Radius



7d: Gross Floor Area Density vs. Radius

Figure 7: Plots of components densities vs. radius

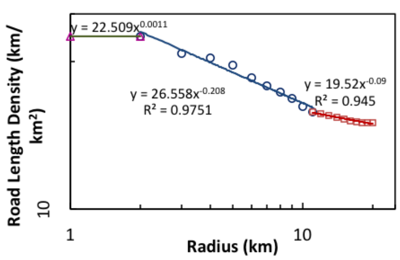
The initial observation is that all the above plots seem to exhibit power law properties, though not at the same level. As mentioned, the presence of a power law is required to prove the presence of fractal characteristics.

The road length density and intersection density plots demonstrate similar patterns, both following a downward trend at a slow pace. This means that the percentage changes in the densities become less and less sensitive to the changes in the radius.

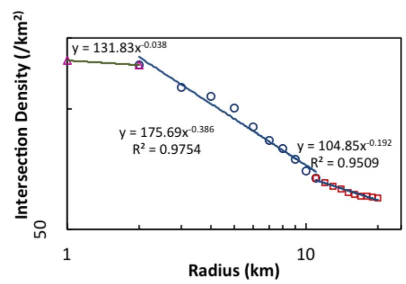
Furthermore, the population+employment density as well as the gross floor area density figures seem to show even more similar patterns. Indeed, from figures 7c and 7d, density values for them are expectedly high at the center (i.e. small radius). Density then drops sharply as one moves away from the center. Moreover, the rates of change of the slopes for both plots gradually start decreasing, demonstrating the reduced sensitivity of the population+employment and the gross floor area densities with respect to the distance from the center. This is also expected; because the further away from the center, the less percentage change in the radius.

Comparing the four figures, one can conclude that the concentration of road segments and intersections within this urban system is less sensitive to the distance from the center as compared to the population+employment and gross floor area. The reason could be the fact that roads and intersections are constrained to a planar expansion, while the population+employment and gross floor area can expand in the three dimensional space, something which will be discussed more later.

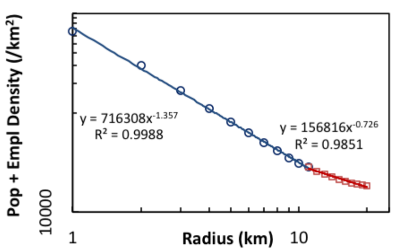
To further study the potential fractal characteristics of these four components, and to determine their fractal dimensions, the same data are redrawn on log-log plots, that transform power laws into straight lines (Figure 8).



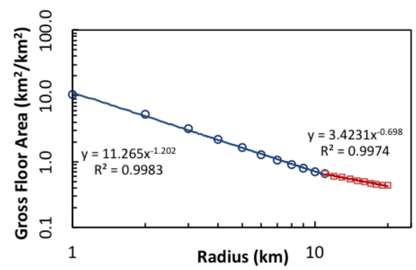
8a: Road length Density vs. Radius



8b: Intersections Density vs. Radius



8c: Population + Employment Density vs. Radius



8d: Gross Floor Area Density vs. Radius

Figure 8: Log-log plots of components densities vs. radius

As a mega city, Chicago has absorbed many smaller urban areas within itself during its evolution history, and therefore changes were expected in the patterns observed over the selected range of the radii, i.e. 21 km. The log-log plots indeed confirmed this expectation, as there are break points in the linear trends in the plots, which one can observe in Figure 8. Nevertheless, using the piece-wise linear patterns observed in the log-log plots, the fractal dimension values for the chosen components of the city were extracted. Also, statistical analyses were performed on the data and the results are presented in Table 2. The R^2 and t-stat values show that all the results are statistically significant.

Component	Radius (km)	Fractal dimension	Std.Dev.	R^2	|t-stat|	Significant?
Road Length	3 to 10	1.21	0.02	0.98	13	Yes
	11 to 21	1.09	0.01	0.95	11	Yes
Intersection	3 to 10	1.39	0.03	0.98	15	Yes
	11 to 21	1.19	0.01	0.95	13	Yes
Population + Employment	3 to 10	1.36	0.02	0.99	77	Yes
	11 to 21	1.73	0.03	0.98	22	Yes
Gross Floor Area	3 to 10	1.20	0.02	0.99	66	Yes
	11 to 21	1.70	0.02	0.99	45	Yes

Table 2: Fractal Dimensions of components densities

4.2 Analysis

In Figure 8, the log-log plots of both the road length and intersection densities show mild linear relationships with respect to the radius. Although there are still visible linear trends in both diagrams, they each show three parts with three different slopes. The first parts of both plots only consist of the first two points, which are within the first 2 km radius from the center. A first plausible explanation is that within 1-2 km radii, areas are small and the presence of a large park (i.e. Grant Park) affects the road density. A second plausible explanation favors the idea that, unlike the rest of the city, strong top-down planning decisions, as opposed to self-organization, were taken in that area due to its commercial importance. A third, and perhaps more likely, explanation points to the fact that the road system may have reached a point of saturation. In other words, Chicago has reached its full horizontal capacity during its evolution, and its expansion has had to switch almost completely from horizontal to vertical direction (e.g. by building skyscrapers). On the other hand, road networks, unlike buildings, are restricted to two dimensions. Due to that, there could be no more new intersections or roads to build.

In contrast, both the log-log plots of population+employment and gross floor area densities show very well-defined linear relationships that start from the center and moves outwards up to the radius of 10 km, after which the slopes of both curves change. Interestingly, the location of the change, which appears in all log-log diagrams, corresponds to the boundary of old

Chicago City with old Cicero Township, which are now completely merged. This fact explains the sudden, yet similar, changes in the trends of all log-log plots in Figure 8. Moreover, the diagrams show that the rates of change of both densities have slowed down beyond the 10-km radius, which reflect the fact that after that point the sensitivities of the population+employment and gross floor area densities with respect to the distance to the center of the city have fallen, i.e. people and businesses are less reactive to a slight change in the distance to the downtown area. Nevertheless, the linear trends of both diagrams continue after the 10 km radius, though with different slopes.

Returning to Figure 8, other than the first parts of both log-log diagrams of the road length and intersection densities, the rest of their plots show patterns similar to each other, as well as to the log-log plots of the population+employment and gross floor area densities, in that they follow two distinct linear trends with a separation point at a radius of around 10 km, thus capturing the hidden boundary between the old Chicago city and the old Cicero Township. Moreover, the decreasing slopes of the two plots represent the fact that the further one moves away from the center of the city, more infrastructure per capita (but less overall) will be required to accommodate the decreasing population.

In order to further investigate the similarity between the road and Intersection densities on one hand and the population+employment and gross floor area densities on the other hand, the corresponding values were plotted against each other (Figures 9 and 10), which clearly shows linear relationships between them.

As for the road and Intersection densities, the two components show a strong statistical relationship, with R^2 of 0.997, as shown in Figure 9.

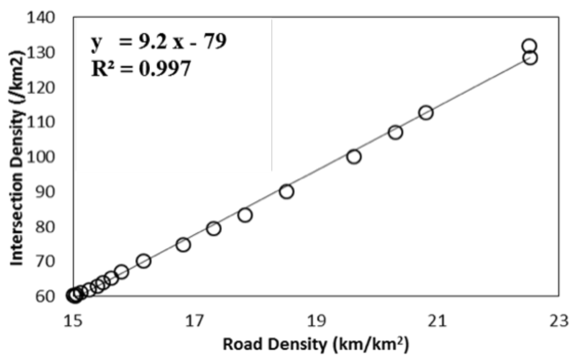


Figure 9: Correlation between Road and Intersection Densities

With respect to the population+employment and gross floor area densities, one would appreciate that their fractal dimensions are very close. This observation points to the high degree of similarity between these two different-in-nature components of the city. In fact, a plot of population+employment versus Gross Floor Area densities show a very good linear relationship between the two components, with a significant R^2 value of 0.997, as shown in Figure 10.

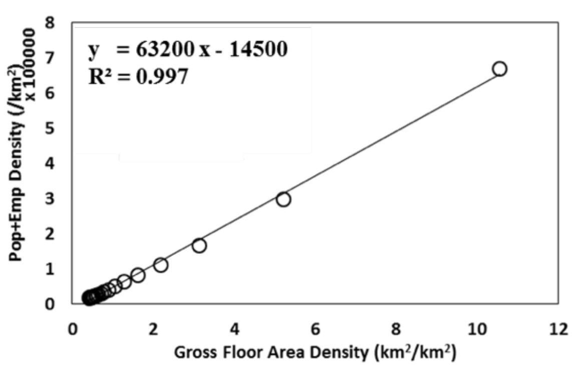


Figure 10: Correlation between Population+Employment and Gross Floor Area Densities

5 Discussion

The trends observed in the road length and intersection densities were similar, and the values obtained for their fractal dimensions are also fairly close. On the other hand, the difference in their fractal dimensions in the 1-10 km range may suggest that the road capacity has been reached (as experienced every day with severe congestion), but further investigation is needed to confirm this hypothesis. That being said, the fractal dimensions calculated for the intersections are slightly and consistently higher than the corresponding fractal dimensions found for the road lengths. An explanation is that by moving away from the center of the city, the average land parcel size increases, therefore leaving less space for roads and intersections. This is particularly true for the number of intersections, since the average block size also tends to increase in suburban areas.

The two components population+employment and building gross floor area showed strong similarities with one another. In fact, their fractal dimensions are close, especially in the 11-21 km range. Although a further investigation is necessary, this result seems to attest the presence of an equilibrium between the *supply* (i.e. buildings) and the *demand* (i.e. population and employment) within that range.

The patterns of all diagrams therefore corresponds to the fact that the further one moves away from the center of an urban system, the less dense it gets in terms or population and employment, building gross floor area, and transportation infrastructure (roads and intersections), something which is expected. Moreover, their rates of change, expectedly, are higher at the beginning but slow down quickly as the distance increases. This, again, is the exhibition of the power law, which is a representative of fractal behavior.

Overall, the results strongly support the hypothesis that the four components considered (population+employment, gross floor area, road length, and intersection densities) in the city of Chicago are fractals in nature, as demonstrated by the presence of power law relationships.

Nevertheless, not all fractals are exactly the same, and as such each possess its own characteristics, including its own fractal dimension. Sudden changes in the behavior of these fractal entities can enable one to identify where the inherent characteristics of a system have changed. This could be a clue to the causes behind such changes, which can then be used to identify the shortcomings or deficiencies of the system.

In the case of the city of Chicago, despite its old history, all of its chosen components show fairly similar patterns. For example, they all show a change in the characteristics of the city at its old boundary. This could be used to identify hidden underlying attributes of an urban system.

6 Conclusion

The work presented in this article offers a simple yet efficient fractal approach to the identification, analysis, and comparison of the characteristics of the components of an urban system. The proposed ring-buffer method is capable of exposing the hidden features of a city, even an old and diverse city such as Chicago with its unique physical and topological barriers.

The study was able to achieve its objectives, namely: to analyze the characteristics of several urban system components, i.e. roads, intersections, population+employment, and gross floor area, of Chicago using the proposed fractal approach; to determine the similarities and differences between the fractal representations of those components; and to explore and explain the reasons behind such similarities and differences.

A further expansion of this work could be the application of the method used in this study to other cities, so that the results can be compared and analyzed and the findings can be used to improve the quality of the observations and conclusions from this fractal approach. As a starter, the same method could also be tested on polycentric cities.

Further work could also focus on the inclusion of more components of urban systems, such as their utility networks (gas, electricity, water, wastewater, etc.). It is expected that these components will also exhibit a fractal nature.

Moreover, the proposed method can be used to explore the evolution of urban complex systems through time, i.e. temporal analysis, which could help in following their evolutionary paths and analyze the impacts of natural or man-made events on how they are shaped today.

Overall, this approach could potentially be used towards a better understanding of how a city, as a complex system, works and how its intertwined components can be studied and improved.

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