Incorporating advanced behavioral models in integer optimization

Michel Bierlaire Shadi Sharif Azadeh

Transport and Mobility Laboratory
School of Architecture, Civil and Environmental Engineering
Ecole Polytechnique Fédérale de Lausanne

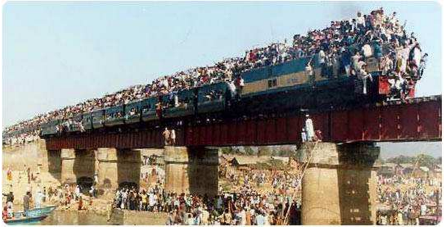
August 26, 2015

Outline

Demand models

- Supply = infrastructure
- Demand = behavior, choices
- Congestion = mismatch

Demand models



- Usually in OR:
- optimization of the supply
- for a given (fixed) demand

Aggregate demand

- Homogeneous population
- Identical behavior
- Price (P) and quantity (Q)
- Demand functions: $P = f(Q)$
- Inverse demand: $Q = f^{-1}(P)$

TRANSP-OR

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Disaggregate demand

- Heterogeneous population
- Different behaviors
- Many variables:
  - Attributes: price, travel time, reliability, frequency, etc.
  - Characteristics: age, income, education, etc.
- Complex demand/inverse demand functions.

Demand-supply interactions

Operations Research

- Given the demand...
- configure the system

Johnson City Enterprise.

Published Every Saturday,

$1. per year—Advance Payment.

SATURDAY, APRIL 7, 1883.

TIME TABLE

E. T., V. & G. R. R.

PASSENGER,	ARRIVES,
No. 1, West,	6:37, a. m.
No. 2, East,	9:45, p. m.
No. 3, West,	11:51, p.m.
No. 4, East,	3:56, a. m.
LOCAL FREIGHT,	ARRIVES,
No. 5,	7:20, a. m.
No. 8,	6:20, p. m.

JNO. W. EAKIN, Agent.

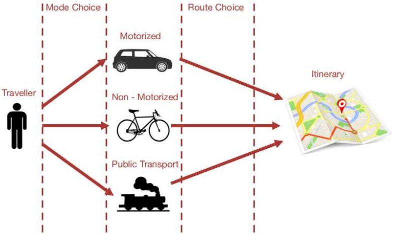
E. T. & W. N. C. R. R.

Passenger, leaves,	7, a. m.
" arrives,	6, p. m.

J. C. HARDIN, Agent.

Behavioral models

- Given the configuration of the system...
- predict the demand



Demand-supply interactions

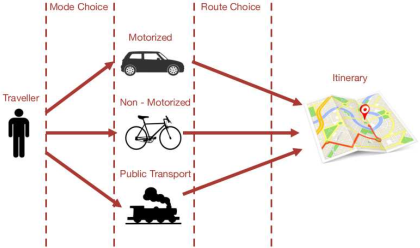
Multi-objective optimization

Minimize costs

Maximize satisfaction

Outline

Measuring satisfaction



Behavioral models

- Demand = sequence of choices
- Choosing means trade-offs
- In practice: derive trade-offs from choice models

Choice models

Theoretical foundations

- Random utility theory
- Choice set: $\mathcal{C}_n$
- $y_{in} = 1$ if $i \in \mathcal{C}_n$, 0 if not
- Logit model:

$$P(i|\mathcal{C}_n) = \frac{y_{in} e^{V_{in}}}{\sum_{j \in \mathcal{C}} y_{jn} e^{V_{jn}}}$$

2000

Decision rules

Neoclassical economic theory

Preference-indifference operator $\gtrsim$

1. reflexivity

$$a \gtrsim a \quad \forall a \in \mathcal{C}_n$$

2. transitivity

$$a \gtrsim b \text{ and } b \gtrsim c \Rightarrow a \gtrsim c \quad \forall a, b, c \in \mathcal{C}_n$$

3. comparability

$$a \gtrsim b \text{ or } b \gtrsim a \quad \forall a, b \in \mathcal{C}_n$$

Decision rules

Utility

$$\exists\ U_n : \mathcal{C}_n \longrightarrow \mathbb{R} : a \rightsquigarrow U_n(a) \text{ such that}$$

$$a \gtrsim b \Leftrightarrow U_n(a) \geq U_n(b) \quad \forall a, b \in \mathcal{C}_n$$

Remarks

- Utility is a latent concept
- It cannot be directly observed

Example

Two transportation modes

$$
\begin{aligned}
U_1 &= -\beta t_1 - \gamma c_1 \\
U_2 &= -\beta t_2 - \gamma c_2
\end{aligned}
$$

with β, $\gamma > 0$

$$U_1 \geq U_2 \text{ iff } -\beta t_1 - \gamma c_1 \geq -\beta t_2 - \gamma c_2$$

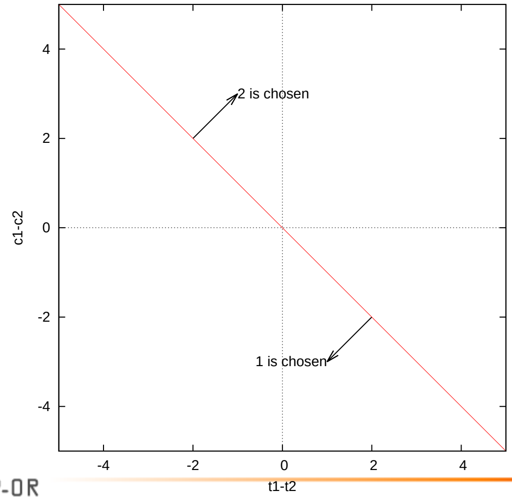
that is

$$-\frac{\beta}{\gamma} t_1 - c_1 \geq -\frac{\beta}{\gamma} t_2 - c_2$$

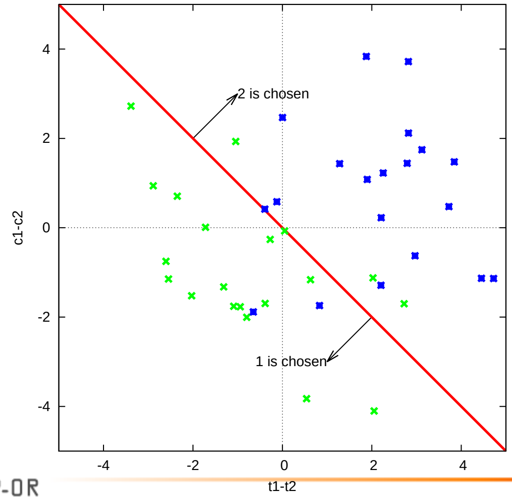
or

$$c_1 - c_2 \leq -\frac{\beta}{\gamma}(t_1 - t_2)$$

Example



Example



Assumptions

Decision-maker

- perfect discriminating capability
- full rationality
- permanent consistency

Analyst

- knowledge of all attributes
- perfect knowledge of $\gtrsim$ (or $U_n(\cdot)$)
- no measurement error

Must deal with uncertainty

- Random utility models
- For each individual n and alternative i

$$U_{in} = V_{in} + \varepsilon_{in}$$

and

$$P(i|\mathcal{C}_n) = P[U_{in} = \max_{j \in \mathcal{C}_n} U_{jn}] = P(U_{in} \geq U_{jn} \; \forall j \in \mathcal{C}_n)$$

Logit model

Utility

$$U_{in} = V_{in} + \varepsilon_{in}$$

Choice probability

$$P_n(i|\mathcal{C}_n) = \frac{y_{in} e^{V_{in}}}{\sum_{j \in \mathcal{C}} y_{jn} e^{V_{jn}}}.$$

- Decision-maker n
- Alternative $i \in \mathcal{C}_n$

Variables: $x_{in} = (z_{in}, s_n)$

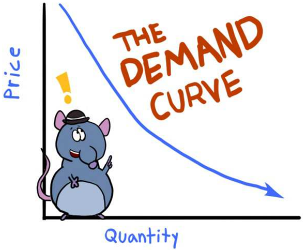
Attributes of alternative i: z_{in}

- Cost / price
- Travel time
- Waiting time
- Level of comfort
- Number of transfers
- Late/early arrival
- etc.

Characteristics of decision-maker n: s_n

- Income
- Age
- Sex
- Trip purpose
- Car ownership
- Education
- Profession
- etc.

Demand curve



Disaggregate model

$$P_n(i|c_{in}, z_{in}, s_n)$$

Total demand

$$D(i) = \sum_n P_n(i|c_{in}, z_{in}, s_n)$$

Difficulty

Non linear and non convex in c_{in} and z_{in}

Outline

A simple example

Data

- $\mathcal{C}$: set of movies
- Population of N individuals
- Utility function:
  $U_{in} = \beta_{in} p_{in} + f(z_{in}) + \varepsilon_{in}$

Decision variables

- What movies to propose? y_i
- What price? p_{in}

Demand model

Logit model

Probability that n chooses movie i:

$$P(i|y, p_n, z_n) = \frac{y_i e^{\beta_{in} p_{in} + f(z_{in})}}{\sum_j y_j e^{\beta_{jn} p_{jn} + f(z_{jn})}}$$

Total revenue:

$$\sum_{i \in C} y_i \sum_{n=1}^{N} p_{in} P(i|y, p_n, z_n)$$

Non linear and non convex in the decision variables

The main idea

The main idea

Linearization

Hopeless to linearize the logit formula (we tried...)

First principles

Each customer solves an optimization problem

Solution

Use the utility and not the probability

A linear formulation

Utility function

$$U_{in} = V_{in} + \varepsilon_{in} = \sum_k \beta_k x_{ink} + f(z_{in}) + \varepsilon_{in}.$$

Simulation

- Assume a distribution for ε_{in}
- E.g. logit: i.i.d. extreme value
- Draw R realizations ξ_{inr}, $r = 1, \ldots, R$
- The choice problem becomes deterministic

Scenarios

Draws

- Draw R realizations ξ_{inr}, $r = 1, \ldots, R$
- We obtain R scenarios

$$U_{inr} = \sum_k \beta_k x_{ink} + f(z_{in}) + \xi_{inr}.$$

- For each scenario r, we can identify the largest utility.
- It corresponds to the chosen alternative.

Comparing utilities

Variables

$$\mu_{ijnr} = \begin{cases} 1 & \text{if } U_{inr} \geq U_{jnr}, \\ 0 & \text{if } U_{inr} < U_{jnr}. \end{cases}$$

Constraints

$$(\mu_{ijnr} - 1)M_{nr} \leq U_{inr} - U_{jnr} \leq \mu_{ijnr} M_{nr}, \forall i, j, n, r.$$

where

$$|U_{inr} - U_{jnr}| \leq M_{nr}, \forall i, j,$$

Comparing utilities

Constraints: $\mu_{ijnr} = 1$

$$0 \leq U_{inr} - U_{jnr} \leq M_{nr}, \forall i, j, n, r.$$

$$U_{jnr} \leq U_{inr}, \forall i, j, n, r.$$

Constraints: $\mu_{ijnr} = 0$

$$-M_{nr} \leq U_{inr} - U_{jnr} \leq 0, \forall i, j, n, r.$$

$$U_{inr} \geq U_{jnr}, \forall i, j, n, r.$$

Accounting for availabilities

Motivation

- If $y_i = 0$, alternative i is not available.
- Its utility should not be involved in any constraint.

New variables: two alternatives are both available

$$\eta_{ij} = y_i y_j$$

Linearization:

$$y_i + y_j \leq 1 + \eta_{ij},$$
$$\eta_{ij} \leq y_i,$$
$$\eta_{ij} \leq y_j.$$

Comparing utilities of available alternatives

Constraints

$$M_{nr}\eta_{ij} - 2M_{nr} \leq U_{inr} - U_{jnr} - M_{nr}\mu_{ijnr} \leq (1-\eta_{ij})M_{nr}, \forall i,j,n,r.$$

$\eta_{ij} = 1$ and $\mu_{ijnr} = 1$

$$0 \leq U_{inr} - U_{jnr} \leq M_{nr}, \forall i,j,n,r.$$

$\eta_{ij} = 1$ and $\mu_{ijnr} = 0$

$$-M_{nr} \leq U_{inr} - U_{jnr} \leq 0, \forall i,j,n,r.$$

Comparing utilities of available alternatives

Constraints

$$M_{nr}\eta_{ij} - 2M_{nr} \leq U_{inr} - U_{jnr} - M_{nr}\mu_{ijnr} \leq (1-\eta_{ij})M_{nr}, \forall i,j,n,r.$$

$\eta_{ij} = 0$ and $\mu_{ijnr} = 1$

$$-M_{nr} \leq U_{inr} - U_{jnr} \leq 2M_{nr}, \forall i,j,n,r,$$

$\eta_{ij} = 0$ and $\mu_{ijnr} = 0$

$$-2M_{nr} \leq U_{inr} - U_{jnr} \leq M_{nr}, \forall i,j,n,r,$$

Comparing utilities of available alternatives

Valid inequalities

$$\begin{aligned} \mu_{ijnr} &\leq y_i, & \forall i, j, n, r, \\ \mu_{ijnr} + \mu_{jinr} &\leq 1, & \forall i, j, n, r. \end{aligned}$$

The choice

Variables

$$w_{inr} = \begin{cases} 1 & \text{if } n \text{ chooses } i \text{ in scenario } r, \\ 0 & \text{otherwise} \end{cases}$$

Maximum utility

$$w_{inr} \leq \mu_{ijnr}, \forall i, j, n, r.$$

Availability

$$w_{inr} \leq y_i, \forall i, n, r.$$

The choice

One choice

$$\sum_{i \in \mathcal{C}} w_{inr} = 1, \forall n, r.$$

Demand and revenues

Demand

$$D_i = \frac{1}{R} \sum_{n=1}^{n} \sum_{r=1}^{R} w_{inr}.$$

Revenues

$$R_i = \frac{1}{R} \sum_{n=1}^{N} p_{in} \sum_{r=1}^{R} w_{inr}.$$

Revenues

Non linear specification

$$R_i = \frac{1}{R}\sum_{n=1}^{N} p_{in} \sum_{r=1}^{R} w_{inr}.$$

Predetermined price levels

Price levels: p_{in}^{ℓ}, $\ell = 1, \ldots, L_{in}$

$$p_{in} = \sum_{\ell=1}^{L_{in}} \lambda_{in\ell} p_{in}^{\ell}.$$

New decision variables

$\lambda_{in\ell} \in \{0, 1\}$

$$\sum_{\ell=1}^{L_{in}} \lambda_{in\ell} = 1.$$

Revenues

Non linear function

$$R_i = \frac{1}{R} \sum_{n=1}^{N} \sum_{\ell=1}^{L_{in}} \lambda_{in\ell} p_{in}^{\ell} \sum_{r=1}^{R} w_{inr}.$$

Linearization

$$\alpha_{inr\ell} = \lambda_{in\ell} w_{inr}$$

Linear specification of revenues

$$R_i = \frac{1}{R} \sum_{n=1}^{N} \sum_{r=1}^{R} \sum_{\ell=1}^{L_{in}} \alpha_{inr\ell} p_{in}^{\ell},$$

with

$$\lambda_{in\ell} + w_{inr} \leq 1 + \alpha_{inr\ell}, \forall i, n, r, \ell,$$
$$\alpha_{inr\ell} \leq \lambda_{in\ell}, \forall i, n, r, \ell,$$
$$\alpha_{inr\ell} \leq w_{inr}, \forall i, n, r, \ell.$$

Back to the example: pricing

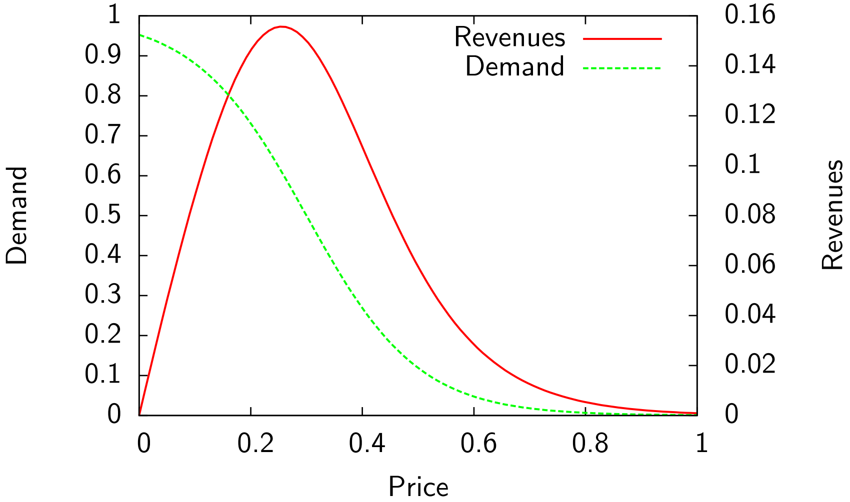
Data

- Two alternatives: my theater (m) and the competition (c)
- We assume an homogeneous population of N individuals

$$U_c = 0 + \varepsilon_c$$
$$U_m = \beta_c p_m + \varepsilon_m$$

- $\beta_c < 0$
- Logit model: ε_m i.i.d. EV

Demand and revenues



Optimization (with GLPK)

Data

- $N = 1$
- $R = 100$
- $U_m = -10p_m + 3$
- Prices: 0.10, 0.20, 0.30, 0.40, 0.50

Results

- Optimum price: 0.3
- Demand: 56%
- Revenues: 0.168

Heterogeneous population

Two groups in the population

$$U_{in} = -\beta_n p_i + c_n$$

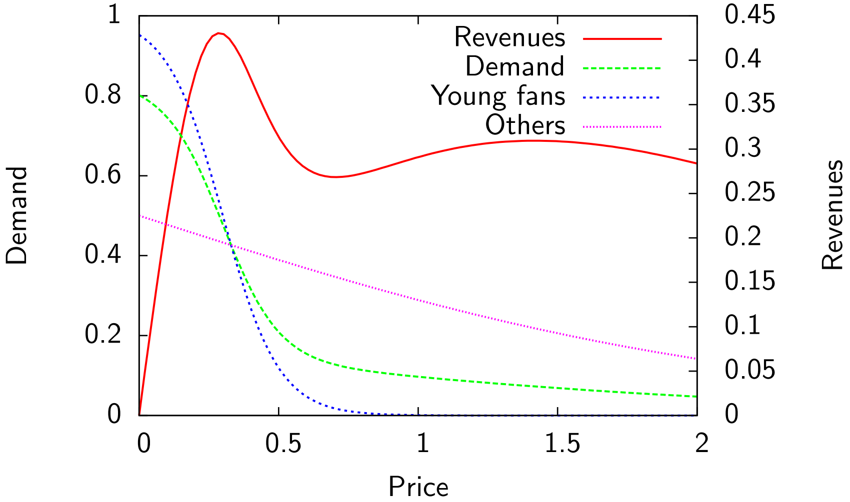
Young fans: 2/3

$\beta_1 = -10, \; c_1 = 3$

Others: 1/3

$\beta_1 = -0.9, \; c_1 = 0$

Demand and revenues



Optimization

Data

- $N = 3$
- $R = 100$
- $U_{m1} = -10p_m + 3$
- $U_{m2} = -0.9p_m$
- Prices: 0.3, 0.7, 1.1, 1.5, 1.9

Results

- Optimum price: 0.3
- Customer 1 (fan): 60% [theory: 50 %]
- Customer 2 (fan) : 49% [theory: 50 %]
- Customer 3 (other) : 45% [theory: 43 %]
- Demand: 1.54 (51%)
- Revenues: 0.48

Two theaters, different types of films

Two theaters, different types of films

Theater m

- Expensive
- Star Wars Episode VII

Theater k

- Cheap
- Tinker Tailor Soldier Spy

Heterogeneous demand

- Two third of the population is young (price sensitive)
- One third of the population is old (less price sensitive)

Two theaters, different types of films

Data

- Theaters m and k
- $N = 6$
- $R = 10$
- $U_{mn} = -10p_m + 4$, $n = 1, 2, 4, 5$
- $U_{mn} = -0.9p_m$, $n = 3, 6$
- $U_{kn} = -10p_k + 0$, $n = 1, 2, 4, 5$
- $U_{kn} = -0.9p_k$, $n = 3, 6$
- Prices m: 1.0, 1.2, 1.4, 1.6, 1.8
- Prices k: half price

Theater m

- Optimum price m: 1.6
- 4 young customers: 0
- 2 old customers: 0.5
- Demand: 0.5 (8.3%)
- Revenues: 0.8

Theater k

- Optimum price m: 0.5
- Young customers: 0.8
- Old customers: 1.5
- Demand: 2.3 (38%)
- Revenues: 1.15

Two theaters, same type of films

Theater m

- Expensive
- Star Wars Episode VII

Theater k

- Cheap
- Star Wars Episode VIII

Heterogeneous demand

- Two third of the population is young (price sensitive)
- One third of the population is old (less price sensitive)

Two theaters, same type of films

Data

- Theaters m and k
- $N = 6$
- $R = 10$
- $U_{mn} = -10p_m + 4$, $n = 1, 2, 4, 5$
- $U_{mn} = -0.9p_m$, $n = 3, 6$
- $U_{kn} = -10p_k + 4$, $n = 1, 2, 4, 5$
- $U_{kn} = -0.9p_k$, $n = 3, 6$
- Prices m: 1.0, 1.2, 1.4, 1.6, 1.8
- Prices k: half price

Theater m

- Optimum price m: 1.8
- Young customers: 0
- Old customers: 1.9
- Demand: 1.9 (31.7%)
- Revenues: 3.42

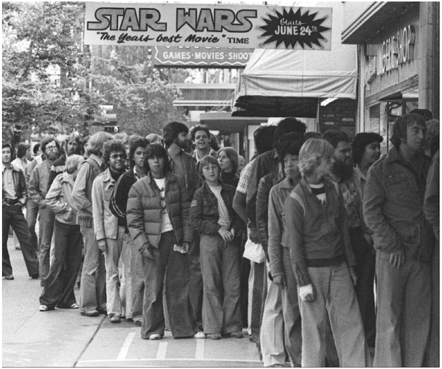
Theater k

Closed

Outline

Dealing with capacities

- Demand may exceed supply
- Not every choice can be accommodated
- Difficulty: who has access?
- Assumption: priority list is exogenous



Priority list

Application dependent

- First in, first out
- Frequent travelers
- Subscribers
- ...

In this framework

The list of customers must be sorted

Dealing with capacities

Variables

- y_{in}: decision of the operator
- y_{inr}: availability

Constraints

$$\sum_{n=1}^{N} w_{inr} \leq c_i$$

$$y_{inr} \leq y_{in}$$

$$y_{i(n+1)r} \leq y_{inr}$$

Constraints

$$c_i(1 - y_{inr}) \leq \sum_{m=1}^{n-1} w_{imr} + (1 - y_{in})c_{\max}$$

$y_{in} = 1,\ y_{inr} = 1$

$$0 \leq \sum_{m=1}^{n-1} w_{imr}$$

$y_{in} = 1,\ y_{inr} = 0$

$$c_i \leq \sum_{m=1}^{n-1} w_{imr}$$

$y_{in} = 0,\ y_{inr} = 0$

$$c_i \leq \sum_{m=1}^{n-1} w_{imr} + c_{\max}$$

Constraints

$$\sum_{m=1}^{n-1} w_{imr} + (1 - y_{in})c_{\max} \leq (c_i - 1)y_{inr} + \max(n, c_{\max})(1 - y_{inr})$$

$y_{in} = 1,\ y_{inr} = 1$

$$1 + \sum_{m=1}^{n-1} w_{imr} \leq c_i$$

$y_{in} = 1,\ y_{inr} = 0$

$$\sum_{m=1}^{n-1} w_{imr} \leq \max(n, c_{\max})$$

$y_{in} = 0,\ y_{inr} = 0$

$$\sum_{m=1}^{n-1} w_{imr} + c_{\max} \leq \max(n, c_{\max})$$

Two theaters, different types of films

Data

- Theaters m and k
- Capacity: 2
- $N = 6$
- $R = 5$
- $U_{mn} = -10p_m + 4, \ n = 1, 2, 4, 5$
- $U_{mn} = -0.9p_m, \ n = 3, 6$
- $U_{kn} = -10p_k + 0, \ n = 1, 2, 4, 5$
- $U_{kn} = -0.9p_k, \ n = 3, 6$
- Prices m: 1.0, 1.2, 1.4, 1.6, 1.8
- Prices k: half price

Theater m

- Optimum price m: 1.8
- Demand: 0.2 (3.3%)
- Revenues: 0.36

Theater k

- Optimum price m: 0.5
- Demand: 2 (33.3%)
- Revenues: 1.15

Example of two scenarios

Customer	Choice	Capacity m	Capacity k
1	0	2	2
2	0	2	2
3	k	2	1
4	0	2	1
5	0	2	1
6	k	2	0
Customer	Choice	Capacity m	Capacity k
1	0	2	2
2	k	2	1
3	0	2	1
4	k	2	0
5	0	2	0
6	0	2	0

Two theaters: all prices divided by 2

Data

- Theaters m and k
- Capacity: 2
- $N = 6$
- $R = 5$
- $U_{mn} = -10p_m + 4, \ n = 1, 2, 4, 5$
- $U_{mn} = -0.9p_m, \ n = 3, 6$
- $U_{kn} = -10p_k + 0, \ n = 1, 2, 4, 5$
- $U_{kn} = -0.9p_k, \ n = 3, 6$
- Prices m: 0.5, 0.6, 0.7, 0.8, 0.9
- Prices k: half price

Theater m

- Optimum price m: 0.5
- Demand: 1.4
- Revenues: 0.7

Theater k

- Optimum price m: 0.45
- Demand: 1.6
- Revenues: 0.72

Example of two scenarios

Customer	Choice	Capacity m	Capacity k
1	0	2	2
2	0	2	2
3	0	2	2
4	k	2	1
5	k	2	0
6	0	2	0

Customer	Choice	Capacity m	Capacity k
1	k	2	1
2	k	2	0
3	0	2	0
4	m	1	0
5	0	1	0
6	m	0	0

Outline

Summary

Demand and supply

- Supply: prices and capacity
- Demand: choice of customers
- Interaction between the two

Discrete choice models

- Rich family of behavioral models
- Strong theoretical foundations
- Great deal of concrete applications
- Capture the heterogeneity of behavior
- Probabilistic models

Optimization

Discrete choice models

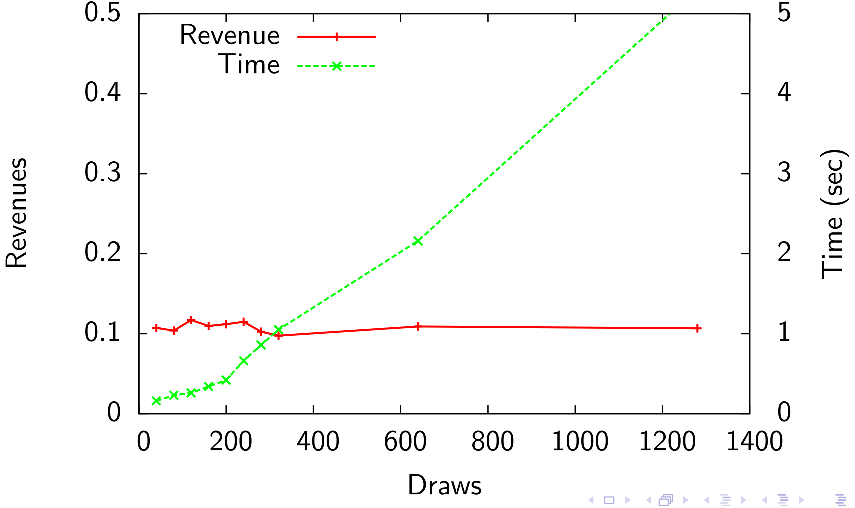
- Non linear and non convex
- Idea: use utility instead of probability
- Rely on simulation to capture stochasticity

Proposed formulation

- Linear in the decision variables
- Large scale
- Fairly general

Number of draws

CPLEX implementation by Shadi



Ongoing research

- Decomposition methods
- Scenarios are (almost) independent from each other (except objective function)
- Individuals are also loosely coupled (except for capacity constraints)

Thank you!

CIRRELT

Université de Montréal

McGill