A Lagrangian relaxation technique for the demand-based benefit maximization problem

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Outline

1 Introduction

2 Demand-based benefit maximization problem

3 Decomposition by operator and customers

4 Decomposition by customers and draws

5 Conclusions and future work
Introduction

Demand-based benefit maximization problem

Decomposition by operator and customers

Decomposition by customers and draws

Conclusions and future work
Discrete choice models and optimization

- Behavioral realism
- Unrealistic assumptions
- Complex formulations

- Tractability
- Linearity and/or convexity
- MILP models
Bridging the gap

- General framework integrating a choice model with an MILP model
- Simulation to linearize the discrete choice model
- Demand-based benefit maximization problem
**Lagrangian relaxation**

**MILP**

\[
Z = \max cx + fy \\
\text{s.t. } Ax \leq b \\
Bx + Dy \leq e \\
Gy \leq h \\
x, y \geq 0 \\
y \text{ integer}
\]

**Lagrangian subproblem**

\[
Z_D(\lambda) = \max cx + fy + \lambda(e - Bx - Dy) \\
\text{s.t. } Bx + Dy \leq e \\
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\]

**Lagrangian dual**

\[
Z_D = \min_{\lambda} Z_D(\lambda)
\]

\[
Z_D(\lambda) \geq Z
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### Lagrangian relaxation

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1. Introduction

2. Demand-based benefit maximization problem

3. Decomposition by operator and customers

4. Decomposition by customers and draws

5. Conclusions and future work
Demand-based benefit maximization problem

Linearization of the choice model

\[ U_{in} = V_{in} + \epsilon_{inr} \]

Draw distribution

\[ U_{inr} = \sum_k \beta_k x_{ink}^e + g_{in}^d(x_{in}^d) + \xi_{inr} \]

\( x^e \) endogenous variable

\( x^d \) exogenous variable (choice)

\[ D_i = \frac{1}{R} \sum_{r=1}^{R} \sum_{n=1}^{N} w_{inr} \]

\[ w_{inr} = \begin{cases} 1 & \text{if } U_{inr} \geq U_{jnr} \forall j \in \mathcal{C}_n, j \neq i \\ 0 & \text{otherwise} \end{cases} \]
Demand-based benefit maximization problem

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Demand-based benefit maximization problem

Benefit maximization problem (1)

- Set of services $\mathcal{C}$ ($i > 0$)
- Population $N$ ($n \geq 1$)
- Price $a_i \leq p_{in} \leq b_{in}$
- Opt-out option $i = 0$
- Capacity $c_i = \sum_{q=1}^{Q} c_{iq} y_{iq}$
- Fixed $f_{iq}$ and variable $v_{iq}$ costs
Benefit maximization problem (2)

Expected gain:

\[ G_i = \sum_{q=1}^{Q} \sum_{n=1}^{N} \sum_{r=1}^{R} \frac{1}{R} \eta_{iqnr} p_{in} w_{iqnr} \]

Operating costs:

\[ C_i = \sum_{q=1}^{Q} \left( f_{iq} + v_{iq} c_{iq} \right) y_{iq} \]
### Benefit maximization problem (3)

<table>
<thead>
<tr>
<th>obj. fun.</th>
<th>$\sum_{i&gt;0} G_i - C_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>availability</td>
<td>operator level and scenario level</td>
</tr>
<tr>
<td>disc. utility</td>
<td>variable capturing availability and utility</td>
</tr>
<tr>
<td>choice</td>
<td>linearization of the highest utility</td>
</tr>
<tr>
<td>price</td>
<td>linearization of the variable $\eta_{iqr}$</td>
</tr>
<tr>
<td>capacity</td>
<td>relation with the availability at scenario level</td>
</tr>
</tbody>
</table>
Computational results

- Parking choices: mixtures of logit model
- \( R = 50 \) draws and \( N = 50 \) customers
- \( |\mathcal{C}| = 3: \) PSP, PUP and FSP (opt-out)
- Experiment testing two approaches:
  1. operator can decide to offer a service
  2. operator is forced to offer all services

<table>
<thead>
<tr>
<th>Approach</th>
<th>Solution time (h)</th>
<th>Capacity</th>
<th>Demand</th>
<th>Prices</th>
<th>Benefit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>PSP</td>
<td>PUP</td>
<td>PSP</td>
<td>PUP</td>
</tr>
<tr>
<td>1</td>
<td>18.7</td>
<td>20</td>
<td>-</td>
<td>19.4</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>33.7</td>
<td>15</td>
<td>5</td>
<td>14.8</td>
<td>4.56</td>
</tr>
</tbody>
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Motivation

- obj. fun.
- availability
- price

- availability
- disc. utility
- choice
- price
- capacity
Lagrangian decomposition

Duplicate $w_{iqnr}$ and relax the copy constraint → Chain of constraints to define new relations → Price specification in the customer subproblem
Lagrangian decomposition

Duplicate $w_{ignr}$ and relax the copy constraint $\rightarrow$ Chain of constraints to define new relations $\rightarrow$ Price specification in the customer subproblem
Duplicate $w_{iqnr}$ and relax the copy constraint

Chain of constraints to define new relations

Price specification in the customer subproblem
Lagrangian decomposition

Duplicate $w_{iqnr}$ and relax the copy constraint → Chain of constraints to define new relations → Price specification in the customer subproblem
Decomposition by operator and customers

Operator subproblem

Capacitated Facility Location Problem (CFLP)

\[
\text{max } Z_0(\theta) = \sum_{i \in C} \sum_{q=1}^{Q} \sum_{n=1}^{N} \sum_{r=1}^{R} \theta_{iqnr} w_{iqnr}' - \sum_{i>0} \sum_{q=1}^{Q} (f_{iq} + v_{iq} c_{iq}) y_{iq}
\]

subject to

\[
\sum_{q=1}^{Q} y_{iq} \leq 1 \quad \forall i > 0 \tag{2}
\]

\[
\sum_{i \in C} \sum_{q=1}^{Q} w_{iqnr}' = 1 \quad \forall n, r \tag{3}
\]

\[
w_{iqnr}' \leq y_{iq} \quad \forall i > 0, q, n, r \tag{4}
\]

\[
w_{iqnr}' = 0 \quad \forall i \notin C, q, n, r \tag{5}
\]

\[
\sum_{n=1}^{N} w_{iqnr}' \leq c_{iq} y_{iq} \quad \forall i > 0, q, r \tag{6}
\]
### Customer subproblem

| obj. fun. | $\sum_{i>0} G_i$ and the relaxed term |
| availability | operator level and scenario level |
| disc. utility | variable capturing availability and utility |
| choice | linearization of the highest utility |
| price | linearization of the variable $\eta_{iqnr}$ |
| capacity | relation with the availability at scenario level |
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Motivation

- Maximization of own utility
- Objective function and capacity constraints
- Behavioral scenario
- Objective function
### Uncapacitated case and revenue maximization

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<th>obj. fun.</th>
<th>$\sum_{i &gt; 0} G_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>availability</td>
<td>no need for discounted utility (no availability)</td>
</tr>
<tr>
<td>utility</td>
<td>linearization of the highest utility</td>
</tr>
<tr>
<td>choice</td>
<td>linearization of the variable $\eta_{iqnr}$</td>
</tr>
<tr>
<td>price</td>
<td></td>
</tr>
<tr>
<td>capacity</td>
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Lagrangian decomposition

⚠️ Price $p_{in}$ is the same across draws ⇒ no decomposition by $n$ and $r$

$$p_{in(r-1)} = p_{inr}$$

$$p_{in1} = p_{in2} = \cdots = p_{inR} = p_{in1}$$

Relaxation with Lagrangian multipliers $\lambda_{inr}$ ⇒ decomposition by $n$ and $r$
Lagrangian decomposition

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Relaxation with Lagrangian multipliers $\lambda_{inr}$ ⇒ decomposition by $n$ and $r$
Subproblem for each $n$ and $r$

\[
\begin{align*}
\text{max} & \quad \sum_{i>0} \left[ \frac{1}{R} \eta_{inr} + (\lambda_{inr} - \lambda_{in(r-1)})p_{inr} \right] \\
\text{subject to} & \quad U_{inr} = \beta_{in} p_{inr} + g_{in}^d(x_{in}^d) + \xi_{inr} \quad \forall i \in \mathcal{C}_n \\
& \quad U_{inr} \leq U_{nr} \quad \forall i \\
& \quad U_{nr} \leq U_{inr} + M_{nr}(1 - w_{inr}) \quad \forall i \\
& \quad \sum_{i \in \mathcal{C}} w_{inr} = 1 \\
& \quad a_{in} w_{inr} \leq \eta_{inr} \quad \forall i > 0 \\
& \quad \eta_{inr} \leq b_{in} w_{inr} \quad \forall i > 0 \\
& \quad p_{inr} - (1 - w_{inr}) b_{in} \leq \eta_{inr} \quad \forall i > 0 \\
& \quad \eta_{inr} \leq p_{inr} - (1 - w_{inr}) a_{in} \quad \forall i > 0
\end{align*}
\]
Decomposition by customers and draws

Algorithm

\[ i = 0 \]

- solve LP (7)–(11)
- if feasible
- choice prices

\[ i = 1 \]

- solve LP (7)–(11)
- if feasible
- choice prices

\[ i = |C| \]

- solve LP (7)–(11)
- if feasible
- choice prices
Subgradient method

Initialization
Number iterations $K$
Initialize $k = 0$
Starting values $\lambda_{inr}^0 = 0$

Subgradients
$s_{inr}^k = p_{in(r-1)}^k - p_{inr}^k$
$s_{inr}^k = 0 \forall i > 0, n, r \Rightarrow $ STOP

Stopping criterion
Increment $k$
If $k = K$, then STOP, else go to Subgradients

Lagrangian multipliers
Compute step size $\gamma^k$
$\lambda_{inr}^{k+1} = \lambda_{inr}^k + \gamma^k s_{inr}^k$
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Conclusions and future work

- Assess the validity of the LR for the uncapacitated case
- Characterize a LR for the capacitated case
- Implement the LR scheme for the customer subproblem
- Test the complete LR with a large number of $N$ and $R$
Conclusions and future work

Questions?

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