Introduction to disaggregate demand models

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Outline

1. Demand models
2. Choice theory
3. Operational model
4. Market shares
5. Willingness to pay
6. Price optimization
7. Summary
Demand models

- Supply = infrastructure
- Demand = behavior, choices
- Congestion = mismatch
Demand models

- Usually in OR:
- optimization of the supply
- for a given (fixed) demand
Aggregate demand

- Homogeneous population
- Identical behavior
- Price \((P)\) and quantity \((Q)\)
- Demand functions: \(P = f(Q)\)
- Inverse demand: \(Q = f^{-1}(P)\)
Disaggregate demand

- Heterogeneous population
- Different behaviors
- Many variables:
  - Attributes: price, travel time, reliability, frequency, etc.
  - Characteristics: age, income, education, etc.
- Complex demand/inverse demand functions.
1. Demand models
2. Choice theory
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Choice

Choice: outcome of a sequential decision-making process

- Definition of the choice problem: How do I get to work?
- Generation of alternatives: car as driver, car as passenger, train
- Evaluation of the attributes of the alternatives: price, time, flexibility, comfort
- Choice: decision rule
- Implementation: travel
A choice theory defines

1. decision maker
2. alternatives
3. attributes of alternatives
4. decision rule
The decision maker is

- an individual or a group of persons.
- If group of persons, we ignore internal interactions.
- Important to capture difference in tastes and decision-making processes.
- Socio-economic characteristics: age, gender, income, education, etc.
Alternatives

Choice set
- Environment: \textit{universal choice set} ($\mathcal{U}$)
- Individual $n$: \textit{choice set} ($\mathcal{C}_n$)

Choice set generation
- Availability
- Awareness

Choice set type
- Continuous
- Discrete
Continuous vs. discrete

Continuous choice set

\[ q_{\text{Milk}} \]

\[ q_{\text{Beer}} \]

\[ p_{\text{Beer}} q_{\text{Beer}} + p_{\text{Milk}} q_{\text{Milk}} = I \]

Discrete choice set

\[ C_n = \{ \text{Car, Bus, Bike} \} \]
Attributes

Describe the item:

- cost
- travel time
- walking time
- comfort
- bus frequency
- etc.
Neoclassical economic theory

Preference-indifference operator $\succeq$

1. **reflexivity**
   
   $$a \succeq a \quad \forall a \in C_n$$

2. **transitivity**

   $$a \succeq b \text{ and } b \succeq c \Rightarrow a \succeq c \quad \forall a, b, c \in C_n$$

3. **comparability**

   $$a \succeq b \text{ or } b \succeq a \quad \forall a, b \in C_n$$
Decision rules

Utility

\[ \exists U_n : C_n \rightarrow \mathbb{R} : a \sim U_n(a) \text{ such that } \]

\[ a \succeq b \iff U_n(a) \geq U_n(b) \quad \forall a, b \in C_n \]

Remarks

- Utility is a latent concept
- It cannot be directly observed
Utility and continuous choice set

Context for the decision maker

- $Q = \{q_1, \ldots, q_L\}$ consumption bundle
- $q_i$ is the quantity of product $i$ consumed
- Utility of the bundle:
  \[ U(q_1, \ldots, q_L) \]
- $Q_a \nsim Q_b$ iff $U(q^a_1, \ldots, q^a_L) \geq U(q^b_1, \ldots, q^b_L)$
- Budget constraint:
  \[ \sum_{i=1}^{L} p_i q_i \leq I. \]
Utility and continuous choice set

Decision-maker solves the optimization problem

$$\max_{q \in \mathbb{R}^L} U(q_1, \ldots, q_L)$$

subject to

$$\sum_{i=1}^{L} p_i q_i = I.$$ 

Example with two products

$$\max_{q_1, q_2} U = \beta_0 q_1^{\beta_1} q_2^{\beta_2}$$

subject to

$$p_1 q_1 + p_2 q_2 = I.$$
Utility and continuous choice set

Example with two products

\[
\max_{q_1, q_2} U = \beta_0 q_1^{\beta_1} q_2^{\beta_2}
\]

subject to

\[
p_1 q_1 + p_2 q_2 = I.
\]

Lagrangian of the problem

\[
L(q_1, q_2, \lambda) = \beta_0 q_1^{\beta_1} q_2^{\beta_2} + \lambda (I - p_1 q_1 - p_2 q_2).
\]

Necessary optimality condition (KKT)

\[
\nabla L(q_1, q_2, \lambda) = 0
\]
Utility and continuous choice set

KKT

\[
\begin{align*}
\beta_0 \beta_1 q_1^{\beta_1 - 1} q_2^{\beta_2} - \lambda p_1 &= 0 \\
\beta_0 \beta_2 q_1^{\beta_1} q_2^{\beta_2 - 1} - \lambda p_2 &= 0 \\
p_1 q_1 + p_2 q_2 - I &= 0.
\end{align*}
\]

We have

\[
\begin{align*}
\beta_0 \beta_1 q_1^{\beta_1 - 1} q_2^{\beta_2} - \lambda p_1 q_1 &= 0 \\
\beta_0 \beta_2 q_1^{\beta_1} q_2^{\beta_2 - 1} - \lambda p_2 q_2 &= 0
\end{align*}
\]

so that

\[
\lambda I = \beta_0 q_1^{\beta_1} q_2^{\beta_2} \left( \beta_1 + \beta_2 \right)
\]

KKT (ctd.)

Therefore

\[
b_0 q_1^{\beta_1} q_2^{\beta_2} = \frac{\lambda I}{(\beta_1 + \beta_2)}
\]

As \( \beta_0 \beta_2 q_1^{\beta_1} q_2^{\beta_2} = \lambda p_2 q_2 \), we obtain

(assuming \( \lambda \neq 0 \))

\[
q_2 = \frac{I \beta_2}{p_2(\beta_1 + \beta_2)}
\]

Similarly, we obtain

\[
q_1 = \frac{I \beta_1}{p_1(\beta_1 + \beta_2)}
\]
Utility and continuous choice set

Demand functions

\[ q_1 = \frac{I\beta_1}{p_1(\beta_1 + \beta_2)} \]

\[ q_2 = \frac{I\beta_2}{p_2(\beta_1 + \beta_2)} \]
Binary optimization

\[
\max_{q \in \{0,1\}^L} U = U(q_1, \ldots, q_L)
\]

with

\[q_i = \begin{cases} 
1 & \text{if product } i \text{ is chosen} \\
0 & \text{otherwise}
\end{cases}
\]

and

\[\sum_{i} q_i = 1.
\]

No optimality condition. Calculus cannot be used anymore.
Utility functions

- Do not work with demand functions anymore
- Work with utility functions
- $U$ is the “global” utility
- Define $U_i$ the utility associated with product $i$.
- It is a function of the attributes of the product (price, quality, etc.)
- We say that product $i$ is chosen if

$$U_i \geq U_j \quad \forall j.$$
Example

Two transportation modes

\[
U_1 = -\beta t_1 - \gamma c_1 \\
U_2 = -\beta t_2 - \gamma c_2
\]

with \( \beta, \gamma > 0 \)

\[
U_1 \geq U_2 \text{ iff } -\beta t_1 - \gamma c_1 \geq -\beta t_2 - \gamma c_2
\]

that is

\[
-\frac{\beta}{\gamma} t_1 - c_1 \geq -\frac{\beta}{\gamma} t_2 - c_2
\]

or

\[
c_1 - c_2 \leq -\frac{\beta}{\gamma} (t_1 - t_2)
\]
Example

![Diagram showing choice theory example with points labeled 1 chosen and 2 chosen. The axes represent $c_1 - c_2$ and $t_1 - t_2$.](image-url)
Example
Assumptions

Decision-maker
- perfect discriminating capability
- full rationality
- permanent consistency

Analyst
- knowledge of all attributes
- perfect knowledge of $\succsim$ (or $U_n(\cdot)$)
- no measurement error

Must deal with uncertainty
- Random utility models
- For each individual $n$ and alternative $i$

\[ U_{in} = V_{in} + \varepsilon_{in} \]

and

\[ P(i|C_n) = P[U_{in} = \max_{j \in C_n} U_{jn}] = P(U_{in} \geq U_{jn} \ \forall j \in C_n) \]
Choice theory

Daniel L. McFadden

- Laureate of *The Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel* 2000
- Owns a farm and vineyard in Napa Valley
- “Farm work clears the mind, and the vineyard is a great place to prove theorems”
Operational model

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Operational model

Assumptions on $V_{in}$

$$V_{in} = V(x_{in}) = V(z_{in}, S_n)$$

Variables
- $z_{in}$: vector of attributes of alternative $i$ for individual $n$
- $S_n$: vector of socio-economic characteristics of $n$
- $x_{in} = (z_{in}, S_n)$

Functional form
- Common assumption: Linear-in-parameter
  $$V_{in} = \sum_p \beta_p (x_{in})_p$$
  - $\beta$: vector of unknown parameters to be estimated
  - Possibility for nonlinear specifications.
Operational model

Assumptions on $\varepsilon_{in}$

- Mean: parameter to be estimated.
- Variance: parameter that cannot be estimated, must be normalized.
- Distribution: two common assumptions

**Probit model**
- Normal distribution
- Motivation: central limit theorem

**Logit model**
- Extreme value distribution
- i.i.d. across $i$ and $n$
- Motivation: Gumbel’s theorem
Logit model

**Utility**

\[ U_{in} = V_{in} + \varepsilon_{in} \]

- Decision-maker \( n \)
- Alternative \( i \in C_n \)

**Choice probability**

\[ P_n(i|C_n) = \frac{e^{V_{in}}}{\sum_{j \in C_n} e^{V_{jn}}} . \]
Model estimation

- Design a sampling protocol
- For each individual $n$ in the sample, and for each alternative $i$,
  - observe the explanatory variables $x_{in}$
  - observe the dependent variable, that is the choice $i_n$.
- Compute the likelihood as a function of the parameters $\beta$:
  \[
  \Lambda_n(\beta) = P_n(i_n|C_n) = \frac{e^{V_{in}}}{\sum_{j\in C_n} e^{V_{jn}}}.
  \]
- Compute the likelihood of the whole sample:
  \[
  \mathcal{L}^*(\beta) = \prod_n \Lambda_n(\beta)
  \]
- Solve the maximum likelihood problem:
  \[
  \hat{\beta} = \arg\max_{\beta} \log \mathcal{L}^*(\beta) = \arg\max_{\beta} \sum_m \log \Lambda_n(\beta).
  \]
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### Aggregation

**Reality**

- Population composed of $N$ individuals
- For each alternative $i$, and each individual $n$, define
  
  $$y_{in} = \begin{cases} 
  1 & \text{if individual } n \text{ chooses alternative } i, \\
  0 & \text{otherwise.} 
  \end{cases}$$

- Total number of individuals selecting item $i$ in the population:
  
  $$N(i) = \sum_{n=1}^{N} y_{in}.$$  

- Market share for item $i$:
  
  $$W(i) = \frac{N(i)}{N} = \sum_{n=1}^{N} \frac{y_{in}}{N}.$$
Forecasting

- Replace $y_{in}$ by $P_n(i|x_n, C_n)$
- Total number of individuals selecting item $i$ in the population:
  \[
  \hat{N}(i) = \sum_{n=1}^{N} P_n(i|x_n, C_n)
  \]
- Market share for item $i$:
  \[
  \hat{W}(i) = \hat{N}(i)/N = \sum_{n=1}^{N} P_n(i|x_n, C_n)/N
  \]
- Issue: no way to access $x_n$ for the entire population.
- Solution: use a representative sample
Demand models

- \( \hat{N}(i) \) or \( \hat{W}(i) \) are the demand models required for OR
- They can be computed for subgroups of the population
- They account for the heterogeneity of
  - behavior,
  - taste,
  - choice contexts.
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7 Summary
Value of time

Objective
- Monetary value of travel time?
- Cost - benefit analysis
- Costs : CHF
- Benefits : travel time savings

Definition
Price that travelers are willing to pay to decrease the travel time.

Motivation
Total time budget is limited, saved time can be used for other activities and, therefore, has value.
Example

Utility functions

\[
U_1 = -\beta t_1 - \gamma c_1 \\
U_2 = -\beta t_2 - \gamma c_2
\]

with \( \beta, \gamma > 0 \)

Choice assumption

\[
U_1 \geq U_2 \text{ if } c_1 - c_2 \leq -\frac{\beta}{\gamma} (t_1 - t_2)
\]

CHF CHF

\frac{\text{CHF}}{\text{hours}} \quad \text{hours}
Value of time

- If utility function is linear
  - the value of time is the ratio between
    - the coefficient of the “time” variable, and
    - the coefficient of the “cost” variable
- Warning: utility is not always linear
- Value of time varies with
  - trip purpose
  - transportation mode
  - trip length
  - income
Example: model choice in Nijmegen

\[ V_{\text{car}} = -0.798 - 0.110 \cdot \text{cost}_{\text{car}} - 1.33 \cdot \text{time}_{\text{car}} \]

\[ V_{\text{train}} = -0.110 \cdot \text{cost}_{\text{train}} - 1.33 \cdot \text{time}_{\text{train}} \]

Value of time \( = -1.33 / -0.110 \approx 12 \text{ euros / h} \approx 0.20 \text{ euros / min} \)

<table>
<thead>
<tr>
<th></th>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>2 h</td>
<td>1.5 h</td>
</tr>
<tr>
<td>Cost</td>
<td>7 €</td>
<td>13 €</td>
</tr>
<tr>
<td>Utility of train</td>
<td>-3.43</td>
<td>-3.43</td>
</tr>
</tbody>
</table>
Other willingness to pay indicators

- Headway (i.e. time between two buses)
- Number of transfers
- Reliability
- etc.

Same methodology:
- The model must involve the corresponding variable
- Willingness-to-pay = ratio between the coefficient of the variable and the cost coefficient

\[ U = -\beta t - \gamma c - \alpha n \]

Willingness-to-pay to have one less transfer: \( \frac{\alpha}{\gamma} \)
Value of time in Switzerland

Reference

Data collection
- Source for recruitment: survey “Kontinuierliche Erhebung zum Personenverkehr” (KEP) by SBB/CFF
- Stated preferences
- Questionnaire designed based on a real reference trip
- Three parts:
  - SP mode choice (car / bus or rail)
  - SP route choice (current mode or alternative mode)
  - Socio-demographics and information about the reference trip
### SP survey

**Mode choice car – rail (main study version)**

<table>
<thead>
<tr>
<th></th>
<th>Car</th>
<th>Rail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Travel costs</td>
<td>18 Fr.</td>
<td>23 Fr.</td>
</tr>
<tr>
<td>Total travel time</td>
<td>40 minutes</td>
<td>30 minutes</td>
</tr>
<tr>
<td>... congested</td>
<td>10 minutes</td>
<td>30 minutes</td>
</tr>
<tr>
<td>... uncongested</td>
<td>30 minutes</td>
<td>30 minutes</td>
</tr>
</tbody>
</table>

Your choice

**Route choice rail (main study version)**

<table>
<thead>
<tr>
<th></th>
<th>Car</th>
<th>Rail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Travel costs</td>
<td>20 Fr.</td>
<td>23 Fr.</td>
</tr>
<tr>
<td>Travel time</td>
<td>40 minutes</td>
<td>30 minutes</td>
</tr>
<tr>
<td>Headway</td>
<td>15 minutes</td>
<td>30 minutes</td>
</tr>
<tr>
<td>No. of changes</td>
<td>1 times</td>
<td>0 times</td>
</tr>
</tbody>
</table>

Your choice
## Data

Number of observations (1225 individuals)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode : car/bus</td>
<td>6</td>
<td>162</td>
<td>186</td>
<td>126</td>
<td>480</td>
</tr>
<tr>
<td>Mode : car/rail</td>
<td>426</td>
<td>1716</td>
<td>2538</td>
<td>1104</td>
<td>5784</td>
</tr>
<tr>
<td>Route : bus for bus users</td>
<td>9</td>
<td>405</td>
<td>450</td>
<td>342</td>
<td>1206</td>
</tr>
<tr>
<td>Route : car for car users</td>
<td>156</td>
<td>846</td>
<td>1176</td>
<td>660</td>
<td>2838</td>
</tr>
<tr>
<td>Route : rail for car users</td>
<td>126</td>
<td>594</td>
<td>837</td>
<td>504</td>
<td>2061</td>
</tr>
<tr>
<td>Route : rail for rail users</td>
<td>324</td>
<td>1008</td>
<td>1881</td>
<td>288</td>
<td>3501</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>1047</strong></td>
<td><strong>4731</strong></td>
<td><strong>7068</strong></td>
<td><strong>3024</strong></td>
<td><strong>15870</strong></td>
</tr>
</tbody>
</table>

In average, $15870/1225 \approx 13$ valid responses out of 16 per respondent.
Willingness to pay

Model specification

Variables

- travel time
- travel cost
- level of congestion (car)
- frequency (TC)
- number of transfers (TC)
- trip length
- income
- inertia
- car availability
- sex
- 1/2-fare CFF
- general subscription
- trip purpose
# Results

<table>
<thead>
<tr>
<th></th>
<th>Business</th>
<th>Commute</th>
<th>Leisure</th>
<th>Shopping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time TC (CHF/h)</td>
<td>49.57</td>
<td>27.81</td>
<td>21.84</td>
<td>17.73</td>
</tr>
<tr>
<td>Time car (CHF/h)</td>
<td>50.23</td>
<td>30.64</td>
<td>29.20</td>
<td>24.32</td>
</tr>
<tr>
<td>Headway (CHF/h)</td>
<td>14.88</td>
<td>11.18</td>
<td>13.38</td>
<td>8.48</td>
</tr>
<tr>
<td>CHF/transfer</td>
<td>7.85</td>
<td>4.89</td>
<td>7.32</td>
<td>3.52</td>
</tr>
</tbody>
</table>
Results
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Choice model captures demand
Demand is elastic to price
Predicted demand varies with price, if it is a variable of the model
In principle, the probability to use/purchase an alternative decreases if the price increases.
The revenue per user increases if the price increases.
Question: what is the optimal price to optimize revenue?

In short
Price↑⇒ profit/passenger↑ and number of passengers ↓
Price↓⇒ profit/passenger↓ and number of passengers ↑
What is the best trade-off?
Revenue calculation

Number of persons choosing alternative $i$ in the population

$$\hat{N}(i) = \sum_{s=1}^{S} N_s P(i|x_s, p_{is})$$

where

- the population is segmented into $S$ homogeneous strata
- $p_{is}$ is the price of item $i$ in segment $s$
- $x_s$ gathers all other variables corresponding to segment $s$
- $P(i|x_s, p_{is})$ is the choice model
- $N_s$ is the number of individuals in segment $s$
Revenue calculation

Total revenue from $i$

$$R_i = \sum_{s=1}^{S} N_s P(i|x_s, p_{is}) p_{is}$$

If the price is constant across segments...

$$R_i = p_i \sum_{s=1}^{S} N_s P(i|x_s, p_i)$$
Price optimization

Optimizing the price of product $i$ is solving the problem

$$\max_{p_i} \sum_{s=1}^{S} N_s P(i|x_s, p_i)$$

Notes:

- It assumes that everything else is equal
- In practice, it is likely that the competition will also adjust the prices
Illustrative example

A binary logit model with

\[ V_1 = \beta_p p_1 - 0.5 \]
\[ V_2 = \beta_p p_2 \]

so that

\[ P(1|p) = \frac{e^{\beta_p p_1 - 0.5}}{e^{\beta_p p_1 - 0.5} + e^{\beta_p p_2}} \]

Two groups in the population:

- Group 1: \( \beta_p = -2, N_s = 600 \)
- Group 2: \( \beta_p = -0.1, N_s = 400 \)

Assume that \( p_2 = 2 \).
Illustrative example
Sensitivity analysis

- Parameters are estimated, we do not know the real value
- 95% confidence interval: $[\hat{\beta}_p - 1.96\sigma, \hat{\beta}_p + 1.96\sigma]$
- Perform a sensitivity analysis for $\beta_p$ in group 2
Sensitivity analysis

\[ \beta = -0.04 \]
\[ \beta = -0.07 \]
\[ \beta = -0.1 \]
\[ \beta = -0.13 \]
\[ \beta = -0.16 \]
Summary

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