# Two-stage column generation

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# Outline

- Introduction
- Motivation
- VRP with Discrete Split Delivery
- Two-stage column generation
- Variable elimination
- Conclusion

# Introduction

### Context

Dantzig-Wolfe (DW) reformulation of combinatorial problems

### Novel idea

Develop a framework in which a combinatorial problem is solved starting from a partial compact formulation, with the same approach used in column generation (CG) for the restricted extensive formulation, obtaining a partial restricted master problem.

The partial restricted master problem is called later on Partial Master Problem (PMP).

# Introduction

**Algorithm 1**: Two-stage column generation

**repeat**
- **repeat**
  - generate variables for partial master problem (CG1)
- **until** *until optimal PMP* ;
- generate variables for partial compact problem (CG2)

**until** *until optimal MP* ;

# Motivation

Many problems exhibit a compact formulation with many variables (possibly an exponential number) which results in an unmanageable associated pricing problem, when the extensive formulation is obtained through DW.

In particular we have identified the following real-world problems:

- TBAP with QC Assignment in maritime container terminals
- VRP with Discrete Split Delivery
- Field Technician Scheduling Problem
- Routing helicopters for crew exchanges on off-shore locations

# TBAP with QC assignment in container terminals

Giallombardo, Moccia, Salani & Vacca (2008)

## Problem description

- Tactical Berth Allocation Plan (TBAP): assignment and scheduling of ships to berths;
- Quay-Cranes (QC) assignment: a QC profile (number of QCs per shift) is assigned to each ship;
- feasible profiles can vary in length (number of shifts dedicated to the ship) and in size (number of QCs dedicated to the ship in each active shift);
- time windows on ship arrival and on berth availabilities.

## Objective

- Maximize the value of chosen profiles.

# VRP with Discrete Split Delivery

Ceselli, Righini & Salani (2007), Nakao & Nagamochi (2007)

## Problem description

- variant of VRP with split delivery;
- each customer demand is represented by one or more orders which consist of a set of items;
- demand can be split (discretized) but items cannot;
- some combinations of items are not allowed because of incompatibilities between items and vehicles, items and locations, etc.

## Objective

- Minimize the total travel costs.

# Field Technician Scheduling Problem

Xu & Chiu (2001)

**Problem description**

- different types of jobs which require different skills;
- each technician is specialized in a field with certain skills;
- time windows on job starting and completion;
- assignment problem (jobs to technicians) + scheduling problem, where the duration of a job depends on the assignment.

**Objective**

- Maximize the number of jobs completed within a time frame.

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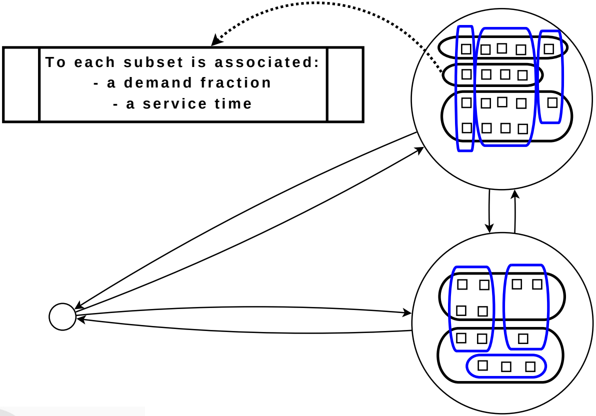
# Crew exchanges on off-shore locations

Sierksma & Tijssen (1998)

# Modeling


To each subset is associated:
- a demand fraction
- a service time

# VRP with Discrete Split Delivery

- $G = (V, E)$ complete graph with $V = \{0\} \cup N$, $(c_{ij}, t_{ij})\ \forall (i,j) \in E$;
- $N$ : set of customers $\{1, \ldots, n\}$;
- $K$ : set of vehicles (capacity $Q$);
- $R$ : set of items; $R = \bigcup_{i \in N} R_i$, $R_i \cap R_j = \emptyset\ \forall i \neq j$, $i, j \in N$;
- $C$ : set of combinations of items; $C = \bigcup_{i \in N} C_i$, $C_i \cap C_j = \emptyset\ \forall i \neq j$, $i, j \in N$;
- $e_c^r$ : 1 if item $r \in R$ is in combination $c \in C$;
- $t_c$ : service time of combination $c \in C$ with $t_c \leq \sum_{r \in c} t_r$, $t_c \geq t_r\ \forall r \in c$;
- $q_c$ : size of combination $c \in C$;
- $[a_i, b_i]$ : time window for customer $i \in N$.

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# VRP with Discrete Split Delivery

## Decision variables

$$x_{ij}^k = \begin{cases} 1 & \text{if arc } (i,j) \in E \text{ is used by vehicle } k \in K; \\ 0 & \text{otherwise.} \end{cases}$$

$$y_c^k = \begin{cases} 1 & \text{if vehicle } k \in K \text{ delivers combination } c \in C; \\ 0 & \text{otherwise.} \end{cases}$$

$T_i^k$ : time when vehicle $k \in K$ arrives at customer $i \in N$

## Constraints

- Flow and precedence constraints
- Demand-satisfaction constraints
- Time-windows constraints
- Capacity constraints

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# VRP with Discrete Split Delivery

## Extensive formulation

$$\min \sum_{p \in P} c_p \lambda_p \tag{1}$$

$$\sum_{p \in P} e_p^r \lambda_p = 1 \qquad \forall r \in R \tag{2}$$

$$\sum_{p \in P} \lambda_p \leq |K| \tag{3}$$

$$\lambda_p \geq 0 \qquad \forall p \in P \tag{4}$$

where:

- $P$ : set of feasible sequences;
- $e_p^r$ : 1 if item $r \in R$ is delivered in sequence $p \in P$ and 0 otherwise;
- $c_p$ : cost of sequence $p \in P$.

# VRP with Discrete Split Delivery

## Pricing problem

$$p^* = \arg\min_{p \in P} \{\tilde{c}_p\} = \arg\min_{p \in P} \{c_p - \sum_{r \in R} \pi_r e_p^r - \pi_0\} \quad (5)$$

## Network formulation

- elementary resource constrained shortest path problem (ERCSPP)
- one node for each variable $y_c$

# Two-stage column generation

## Algorithm

- start with a subset of variables $y_c$ in $C'$ (heuristic);
- compute a bound of the reduced cost for variables $y_c \notin C'$;
- add the most profitable variable;
- iterate.

# Two-stage column generation

## Advantages

- the pricing problem is easier to solve
- possibly many sub-optimal compact variables are left out from the formulation

## Drawbacks

- we don't obtain a valid lower bound

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# VRP with Discrete Split Delivery

## Possible solution to LB computation

Add some ad-hoc artificial variables $y_c^{art}$ to the partial compact formulation.

In particular, we create artificial super-optimal subset of items by combining variables $y_c$ not yet in the partial compact formulation such that:

- they cover the maximum number of items;
- with the minimum service time;
- with the minimum total size.

# Variable elimination

Given an integer linear program (IP) with an upper bound UB, with objective $\min c^T x$ and constraints $Ax \geq b, x \in \mathbb{Z}_+^n$, let $\pi$ be a feasible solution to the dual of the linear programming relaxation of (IP).

**Theorem** (Nemhauser & Wolsey, 1988)

If the reduced cost of a non-negative integer variable exceeds a given optimality gap, the variable must be zero in any optimal integer solution. In other words:

$$\tilde{c}_e = (c - \pi A)_e > UB - \pi b \implies x_e = 0 \tag{6}$$

# Variable elimination

**Theorem** (Irnich et al., 2007)

If the minimum reduced cost of all path variables of a DW master problem containing arc $(i, j)$ exceeds a given optimality gap, no path that contains arc $(i, j)$ can be used in an optimal solution. Hence, the arc $(i, j)$ can be eliminated. In other words:

$$\min_{P \in \mathcal{F}_{ij}^{st}} \tilde{c}_P(\pi) > UB - \pi b \implies x_{ij} = 0 \tag{7}$$

where $\mathcal{F}_{ij}^{st}$ is the set of feasible $s - t$ paths containing arc $(i, j)$.

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# VRP with Discrete Split Delivery

Let the restricted master problem (MP) be defined on the whole set $C$ and let the partial restricted master problem (PMP) be defined on a subset of combinations $C' \subset C$.

Let $UB$ be an upper bound for both MP and PMP, let $LB$ be a lower bound for MP and $LB'$ be a lower bound for PMP, with $LB' \geq LB$.

Let $\pi$ be a feasible dual solution to MP and $\pi'$ be a feasible dual solution to PMP.

**We define the following quantities:**

$$lb_c = LB + \min_{p \in \mathcal{F}_c} \tilde{c}_p = LB + \min_{p \in \mathcal{F}_c} \{c_p - \sum_{r \in R} e_p^r \pi_r^* - \pi_0^*\} \tag{8}$$

$$lb'_c = LB' + \min_{p \in \mathcal{F}'_c} \tilde{c}_p = LB' + \min_{p \in \mathcal{F}'_c} \{c_p - \sum_{r \in R} e_p^r \pi_r'^* - \pi_0'^*\} \tag{9}$$

where:

- $\mathcal{F}_c = \{$ (subset of) feasible sequences in $P = P(C)$ : $y_c = 1\}$
- $\mathcal{F}'_c = \{$ (subset of) feasible sequences in $P = P(C')$ : $y_c = 1\}$

# VRP with Discrete Split Delivery

## Variable elimination

$$lb_c > UB \implies y_c = 0 \text{ in optimal MP (over } C\text{)}$$
$$lb'_c > UB \implies y_c = 0 \text{ in optimal PMP (over } C'\text{)}$$

### Question

*Can variable elimination in PMP be extended to MP?*

| | *PMP* | *MP* |
|---|---|---|
| A | true | true |
| B | true | false |
| C | false | true |
| D | false | false |

# VRP with Discrete Split Delivery

### Facts

- PMP is contained in MP
- $LB' \geq LB$

### Conjecture

$$lb'_c \geq lb_c \tag{10}$$

- Conjecture true $\implies$ (C) never occurs.
- Unfortunately (B) is possible.

# Compact variable generation

**Algorithm 2**: $y_c$ - Compact variable generation

**repeat**
- **repeat**
  - generate variables for partial master problem (CG1)
- **until** *until optimal PMP* ;
- **repeat**
  - $\hat{y}_c = \min_{y_c \in C \setminus C'} \{(c_{ki} + c_{ij} - c_{kj} | k \neq j, c \in C_i) - \sum_r e_c^r \pi_r\}$
  - **if** $\min_{p \in \mathcal{F}'_{C' \cup c}} \{\tilde{c}_p\} < 0$ **then**
    - add column $p$ and $\hat{y}_c$ to $C'$
  - **else**
    - **if** $\min_{p \in \mathcal{F}'_c} \{\tilde{c}_p\} \leq UB - LB$ **then**
      - add $\hat{y}_c$ to $C'$
    - **else**
      - **if** $\min_{p \in \mathcal{F}'_{c \cup y_c^{art}}} \{\tilde{c}_p\} > UB - LB$ **then**
        - fix variable $\hat{y}_c$ to 0
      - **else**
        - add $\hat{y}_c$ to $C'$
      - **end**
    - **end**
  - **end**
- **until** *until some $y_c$ has been added to $C'$ or $C \setminus C' = \emptyset$* ;

**until** *until all $y_c$ are either in $C'$ or fixed to* 0 ;

# Implementation

- Exact algorithm for the Discrete SDVRP based on Branch&Price
- Pricing solved using bi-directional dynamic programming with DSSR
- Estimation of $lb_c^l$ via dynamic programming with DSSR
- Not sophisticated branching rules: vehicles first, arcs next
- No massive pricing, No additional cuts at master level

# Preliminary results (qualitative)

- Instances derived from Solomon data set for the VRPTW (few **manually** generated instances)
- Instances **with** sub-optimal compact variables are correctly detected
- (+) The overall pricing time has been reduced
- (+) The number of generated columns has been reduced as well
- (-) The method is sensitive to initialization
- (?) Even with no eliminated compact variables the overall number of generated columns is smaller

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# Conclusions and Outlook

Two stage column generation:

- Methodology to accelerate an overall B&P algorithm via generation of compact formulation variables
- Useful when the compact formulation exhibits a large number of variables, but **not only**

Outlook:

- Full validation of the model

**CTW** is a great conference, we had feedback **before** the talk!
(apparently) There is a better method to estimate $lb'_c$, see you next year!