Solution methods for an integrated airline schedule planning and revenue management model

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Workshop on Large Scale Optimization

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Motivation

- Demand responsive transportation systems
  - Better representation of demand ⇒ Appropriate demand models
  - Integration of supply-demand interactions in transportation models

Today’s talk:
- A brief description of the integrated model
- A heuristic method
- Transformation of the problem & Generalized Benders Decomposition
Integrated schedule planning and revenue management

Aim: to take better fleeting decisions with the information provided by the demand model
Integrated model - Schedule planning

Max \( \sum_{h \in H} \sum_{s \in S^h} \sum_{i \in (I_s \setminus I'_s)} \left( d_i - \sum_{j \in I_s} t_{i,j} + \sum_{j \in (I_s \setminus I'_s)} t_{j,i} b_{j,i} \right) p_i - \sum_{k \in K} \sum_{f \in F} C_{k,f} x_{k,f} : \text{revenue} - \text{cost} \) \hspace{1cm} (1)

s.t. \( \sum_{k \in K} x_{k,f} = 1 : \text{mandatory flights} \) \hspace{1cm} \forall f \in F^M \hspace{1cm} (2)

\( \sum_{k \in K} x_{k,f} \leq 1 : \text{optional flights} \) \hspace{1cm} \forall f \in F^O \hspace{1cm} (3)

\( y_{k,a,t}^- + \sum_{f \in \text{In}(k,a,t)} x_{k,f} = y_{k,a,t}^+ + \sum_{f \in \text{Out}(k,a,t)} x_{k,f} : \text{flow conservation} \) \hspace{1cm} \forall [k,a,t] \in N \hspace{1cm} (4)

\( \sum_{a \in A} y_{k,a,\text{minE}_a}^- + \sum_{f \in \text{CT}} x_{k,f} \leq R_k : \text{fleet availability} \) \hspace{1cm} \forall k \in K \hspace{1cm} (5)

\( y_{k,a,\text{minE}_a}^- = y_{k,a,\text{maxE}_a}^+ : \text{cyclic schedule} \) \hspace{1cm} \forall k \in K, a \in A \hspace{1cm} (6)

\( \sum_{h \in H} \pi_{k,f}^h \leq Q_k x_{k,f} : \text{seat capacity} \) \hspace{1cm} \forall f \in F, k \in K \hspace{1cm} (7)

\( x_{k,f} \in \{0,1\} \) \hspace{1cm} \forall k \in K, f \in F \hspace{1cm} (8)

\( y_{k,a,t} \geq 0 \) \hspace{1cm} \forall [k,a,t] \in N \hspace{1cm} (9)

- Itinerary-based fleet assignment
- Spill and recapture
Integrated model - Revenue management

\[
\sum_{s \in S^h} \sum_{i \in (I_s \setminus l_s')} \delta_{i,f} d_i - \sum_{j \in I_s} \delta_{i,f} t_{i,j} + \sum_{j \in (I_s \setminus l_s')} \delta_{i,f} t_{j,i} b_{j,i} \leq \sum_{k \in K} \pi_{k,f}^h: \text{ demand-capacity} \quad \forall h \in H, f \in F \tag{10}
\]

\[
\sum_{j \in I_s \setminus i} t_{i,j} \leq d_i: \text{ total spill} \quad \forall h \in H, s \in S^h, i \in (I_s \setminus l_s') \tag{11}
\]

\[
\tilde{d}_i = D_s \sum_{j \in I_s} \frac{\exp(V_i(p_i, z_i, \beta))}{\sum \exp(V_j(p_j, z_j, \beta))}: \text{ logit demand} \quad \forall h \in H, s \in S^h, i \in I_s \tag{12}
\]

\[
b_{i,j} = \frac{\exp(V_j(p_j, z_j, \beta))}{\sum_{k \in I_s \setminus \{i\}} \exp(V_k(p_k, z_k, \beta))}: \text{ recapture ratio} \quad \forall h \in H, s \in S^h, i \in (I_s \setminus l_s'), j \in I_s \tag{13}
\]

\[
d_i \leq \tilde{d}_i: \text{ realized demand} \quad \forall h \in H, s \in S^h, i \in I_s \tag{14}
\]

\[
LB_i \leq p_i \leq UB_i: \text{ bounds on price} \quad \forall h \in H, s \in S^h, i \in I_s \tag{15}
\]

\[
t_{i,j}, b_{i,j} \geq 0 \quad \forall h \in H, s \in S^h, i \in (I_s \setminus l_s'), j \in I_s \tag{16}
\]

\[
\pi_{k,f}^h \geq 0 \quad \forall h \in H, k \in K, f \in F \tag{17}
\]
Heuristic method

Available solvers\(^1\) are able to converge on instances with about 35 flights. We devised a heuristic procedure based on two simplified versions of the model:

- **FAM\(^{LS}\)**: price-inelastic schedule planning model ⇒ MILP
  - Explores new fleet assignment solutions based on a local search
  - Price sampling
  - Variable neighborhood search (VNS)

- **REV\(^{LS}\)**: Revenue management with fixed capacity ⇒ NLP
  - Optimizes the revenue for the explored fleet assignment solution

\(^1\)BONMIN: Bonami et al. (2008), An algorithmic framework for convex mixed integer nonlinear programs. Discrete Optimization, 5(2):186-204
### Heuristic method

**Require:** \( x^0, y^0, d^0, p^0, t^0, b^0, \pi^0, \) time\(_{\text{max}}\), \( n_{\text{min}}, n_{\text{max}} \), notImpr, tabuListSize

\( g := 0, \) time := 0, \( n_{\text{fixed}} := n_{\text{min}}, \) notImpr := 0, \( z^* := -\text{INF}, \) tabuList := 0

repeat

\( p^g := \text{Price sampling}(t^g - 1, p^g - 1, d^g - 1) \)
\( \{d^g, b^g\} := \text{Logit model}(p^g) \)
\( L := \text{Fixing}(x^g - 1, t^g - 1, n_{\text{fixed}}) \)
\( \{x^g, y^g, \pi^g, t^g\} := \text{solve } z_{\text{FAMLS}}(p^g, d^g, b^g, L) \)

if \((\bar{x}^g \notin \text{tabuList})\) then

\( \text{tabuList} := \text{tabuList} \cup x^g \)
\( \{p^g, d^g, b^g, \pi^g, t^g\} := \text{solve } z_{\text{REVLS}}(x^g, y^g) \)

if \((z_{\text{REVLS}} \geq z^*)\) then

Update \( z^* \)
Intensification: \( n_{\text{fixed}} := n_{\text{fixed}} + 1 \) when \( n_{\text{fixed}} < n_{\text{max}} \)
notImpr := 0

else if \((\text{notImpr} == 3)\) then

Diversification: \( n_{\text{fixed}} := n_{\text{fixed}} - 1 \) when \( n_{\text{fixed}} > n_{\text{min}} \)
notImpr := notImpr - 1

end if

end if

\( g := g + 1 \)

until time \( \geq \) time\(_{\text{max}}\)
Local search

- Neighborhood solutions are visited based on the spill rather than a fully random search
- Price sampling:
  - A random price is drawn for each itinerary
  - If the spilled passengers are higher than the average ⇒ decrease the price
  - Otherwise ⇒ increase the price
- Fixing FAM solutions - VNS:
  - The itineraries are sorted according to their spilled number of passengers
  - Low spill value ⇒ associated flights have a higher probability to be fixed to their current aircraft
  - If the solution is improved more assignments are fixed and vice versa.
Performance of the heuristic

The omitted instances are the ones where the sequential approach has the same solution as the integrated model.

<table>
<thead>
<tr>
<th>Best solution reported by BONMIN</th>
<th>Sequential approach (SA)</th>
<th>Heuristic results</th>
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<tbody>
<tr>
<td>Profit</td>
<td>Time (sec)</td>
<td>Profit</td>
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</table>
Log transformation of the problem

\[ \tilde{d}_i = D_s \cdot \frac{\exp(\beta \ln(p_i) + c_i)}{\sum_{j \in I_s} \exp(\beta \ln(p_j) + c_j)} \quad \forall h \in H, s \in S^h, i \in I_s \]

A new variable \( \nu_s \) \(^2\) is defined:

\[ \nu_s = \frac{1}{\sum_{j \in I_s} \exp(\beta \ln(p_j) + c_j)} \]

\[ \text{Prob}_i^s = \nu_s \exp(\beta \ln(p_i) + c_i) \]

\[ \sum_{i \in I_s} \text{Prob}_i^s = 1 \]

\[ \tilde{d}_i = D_s \text{Prob}_i^s \]

\(^2\)As proposed by Cornelia Schön (2008)
Log transformation

The log transformation for the choice probability:

\[ \ln(\text{Prob}_i^s) = \ln(\upsilon_s) + \beta \ln(p_i) + c_i \]

\[ \text{Prob}_i^{s'} = \upsilon'_s + \beta p'_i + c_i, \]

where \( \text{Prob}_i^s > 0, \upsilon_s > 0, p_i > 0 \)

- We get rid off the non-convexity of the demand model...
Transformed revenue model - no spill effects

The fleet assignment decision variables are fixed \((x_{k,f}, y_{k,a,t})\), we have an NLP.

\[
\max \sum_{h \in H} \sum_{s \in S^h} \sum_{i \in (I_s \setminus I_s')} \exp(p'_i + d'_i) \Rightarrow p'_i + d'_i
\]

\[
\text{s.t. } \sum_{s \in S^h} \sum_{i \in (I_s \setminus I_s')} \delta_{i,f} \exp(d'_i) \leq \sum_{k \in K} \pi^h_{k,f} \quad \forall f \in F^*, h \in H
\]

\[
\sum_{h \in H} \pi^h_{k,f} = Q_k X_{k,f} \quad \forall f \in F, k \in K
\]

\[
d'_i \leq \upsilon'_s + \beta p'_i + c_i + \ln D_s \quad \forall h \in H, s \in S^h, i \in I_s
\]

\[
\sum_{i \in I_s} \exp(\upsilon'_s + \beta p'_i + c_i) = 1 \quad \forall h \in H, s \in S^h
\]

\[
\ln(LB_i) \leq p'_i \quad \forall h \in H, s \in S^h, i \in I_s
\]

\[
p'_i \leq \ln(UB_i) \quad \forall h \in H, s \in S^h, i \in I_s
\]

\[
d'_i, p'_i, \upsilon'_s \in \mathbb{R} \quad \forall h \in H, s \in S^h, i \in I_s
\]

\[
\pi^h_{k,f} \geq 0 \quad \forall h \in H, k \in K, f \in F
\]
Master problem - GBD

\[
\begin{align*}
\text{max } & \alpha \\
\text{s.t. } & \alpha \leq \sum_{h \in H} \sum_{s \in S^h} \sum_{i \in (I_s \setminus I_s')} \exp(P'_c + D'_i) - \sum_{k \in K} \sum_{f \in F} C_{k,f} X_{k,f}^c \\
& + \sum_{k \in K} \sum_{f \in F} (Q_k MR_{k,f}^c - C_{k,f})[x_{k,f} - X_{k,f}^c] \quad \forall c \in \text{CUTS} \quad (27) \\
& \sum_{k \in K} x_{k,f} = 1 \quad \forall f \in F^M \quad (28) \\
& \sum_{k \in K} x_{k,f} \leq 1 \quad \forall f \in F^O \quad (29) \\
& y_{k,a,t}^- + \sum_{f \in \text{In}(k,a,t)} x_{k,f} = y_{k,a,t}^+ + \sum_{f \in \text{Out}(k,a,t)} x_{k,f} \quad \forall [k,a,t] \in N \quad (30) \\
& \sum_{a \in A} y_{k,a,\text{minE}_a}^- + \sum_{f \in CT} x_{k,f} \leq R_k \quad \forall k \in K \quad (31) \\
& y_{k,a,\text{minE}_a}^- = y_{k,a,\text{maxE}_a}^+ \quad \forall k \in K, a \in A \quad (32) \\
& x_{k,f} \in \{0,1\} \quad \forall k \in K, f \in F \quad (33) \\
& y_{k,a,t} \geq 0 \quad \forall [k,a,t] \in N \quad (34) \\
\end{align*}
\]

\textbf{Li and Sun 2006 - Generalized Benders Decomposition}
Lagrangian multipliers - Marginal revenue of each seat

\[
\text{MR}_{k,f}^c = \frac{\sum_{h \in H} \prod_{k,f}^h \lambda_{k,f,h}}{Q_k} \quad \forall f \in F, k \in K, X_{k,f}^c = 1
\]

- $\lambda_{k,f,h}$ are the Lagrangian multipliers related to the demand-capacity constraints
- price of one seat at flight $f$ on class $h$ and plane type $k$.
- obtained with the application of the optimality conditions
GBD framework

- Initial FAM solution \((x_{k,f})\)
- Repeat until \(UB \leq LB\)
  - Solve REV subproblem which is an NLP and obtain...
    - price, demand, allocated seats \((p'_i, d'_i, \pi^h_{k,f})\)
    - Lagrangian multipliers \(\Rightarrow\) Benders cuts
    - A lower bound (LB) for the problem
  - Solve the FAM master problem which is a MILP and obtain...
    - an updated FAM solution \((x_{k,f})\)
    - An upper bound (UB) for the problem
A small example

- 2 airports CDG-MRS
- 4 flights - all are mandatory
- 2 aircraft types: 37-50 seats

We start with an initial FAM solution:

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<th>AC1</th>
<th>AC2</th>
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<tr>
<td>F2</td>
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<tr>
<td>F3</td>
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</tr>
<tr>
<td>F4</td>
<td>X</td>
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</table>
A small example - GBD iterations

### Iteration 1

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<tr>
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<th>Sub</th>
<th>Master</th>
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<tbody>
<tr>
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<tr>
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#### Result

12522.8 14822.8  

### Iteration 3

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<td></td>
<td>F4</td>
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#### Result

12696.8 12696.8  

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Motivation

Integrated model

Heuristic

Log transformation & GBD

On-going work
Conclusions and on-going work

- The integrated model has promising results
- ... which motivates the effort in devising solution methodologies

- GBD will be tested
- The spill effects will be added back to the model
Thank you for your attention!