Introduction à l'optimisation et la recherche opérationnelle (2018–2019)

Professeur : Michel Bierlaire, Assistants responsables : Nikola Obrenovic, Nourelhouda Dougui

Algorithme du simplexe (12 octobre 2018)

Question 1:

Consider the following optimization problem :

$$\max_{x \in \mathbb{R}^2} 3x_1 + x_2$$

subject to

$$\begin{aligned} 2x_1 + 2x_2 &\geq 1 \\ x_1 + x_2 &\leq 2 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$

1. Represent the constraints of the problem graphically.
2. Find all the basic solutions and, for each one, mention if it is feasible or not.
3. Find the optimal solution by vertex enumeration.

To invert the 2×2 and 3×3 matrices, use formulas given at :

http ://mathworld.wolfram.com/MatrixInverse.html,

utilize Matlab or use the calculated inversed matrices given in the Appendix.

Question 2:

Consider the optimization problem

$$\min_{x \in \mathbb{R}^2} -x_1 - \frac{3}{2}x_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 7 \\ -0.5x_1 + x_2 &\geq 1 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solve this problem using the graphical method.

Question 3:

Consider the optimization problem

$$\min 2x_1 + 6x_2$$

subject to

$$\begin{aligned} x_1 + 3x_2 &\leq 7 \\ x_1 + x_2 &\geq \frac{3}{2} \\ x_1 &\geq \frac{3}{2} \\ x_2 &\geq 0. \end{aligned}$$

1. List the basic solutions.
2. For each feasible basic solution, find the corresponding reduced costs. What is the solution of the optimization problem ?

To invert the 2×2 and 3×3 matrices, use formulas given at :

http ://mathworld.wolfram.com/MatrixInverse.html,

utilize Matlab or use the calculated inversed matrices given in the Appendix.

Question 4:

Consider the following linear optimization problem

$$\max 4x_1 - 3x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 6 \\ x_1 - x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

1. Represent the feasible region graphically.
2. Write the standard form of the problem.
3. Solve the problem by the algebraic simplex method. At each iteration, specify
   (a) the basic variables ;
   (b) the current iteration (both analytically and graphically) ;
   (c) the basic direction imposed by the algorithm (both analytically and graphically) ;
   (d) the step length along the basic direction.

   Hint : Verify that the point (0, 0) is a basic feasible solution. If so, start the algorithm from this point.

Question 5 – Multiple Choice Questions:

1. Which of the following statements is ***correct*** about the simplex method ?
   (a) Feasibility may not be satisfied at each iteration of simplex method.
   (b) If reduced cost of a decision variable is zero, the corresponding decision variable is always a basic variable.
   (c) The optimal solution is found when all the reduced costs are nonnegative.
2. Consider a linear problem of minimization type. In that case,
   (a) alternate optimal solutions to a linear problem may exist only when the objective function line is parallel to a constraint line.
   (b) if the feasible region is unbounded, then the optimal objective function value is also unbounded.
   (c) if the right hand side value of an inferior constraint is increased, then the optimal objective function value may increase.

Appendix

- The inverse of matrix $B = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$.
- The inverse of matrix $B = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} -1/2 & 0 \\ 1/2 & 1 \end{pmatrix}$.
- The inverse of matrix $B = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$.
- The inverse of matrix $B = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} -1/2 & 0 \\ 1/2 & 1 \end{pmatrix}$.
- The inverse of matrix $B = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -3 & 2 \end{pmatrix}$.
- The inverse of matrix $B = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1/3 & 0 & -1/3 \\ 1/3 & -1 & 2/3 \end{pmatrix}$.
- The inverse of matrix $B = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} -1/2 & 3/2 & 0 \\ 1/2 & -1/2 & 0 \\ -1/2 & 3/2 & -1 \end{pmatrix}$.
- The inverse of matrix $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.
- The inverse of matrix $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$.
- The inverse of matrix $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$.

— The inverse of matrix $B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

— The inverse of matrix $B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 1/3 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

— The inverse of matrix $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is $B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.