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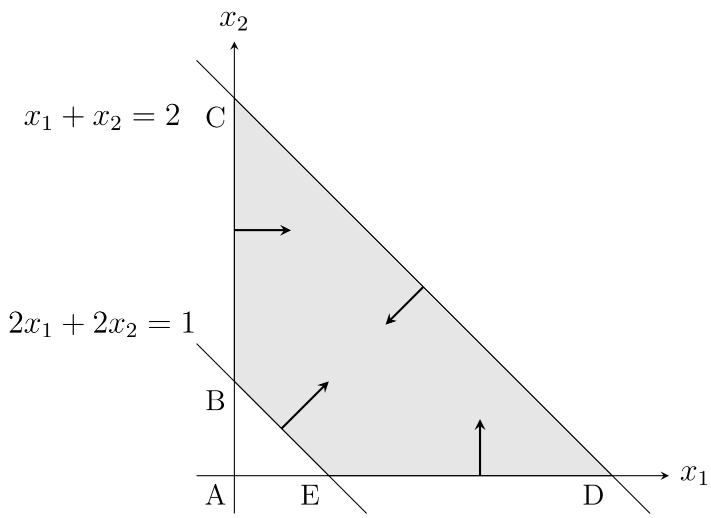
Algorithme du simplexe – corrigé (12 octobre 2018)

Solution of the question 1:

1. The polygon is :

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} | 2x_1 + 2x_2 \geq 1, x_1 + x_2 \leq 2, x_1 \geq 0, x_2 \geq 0 \right\}.$$

   We plot it in the following graph :



2. Let us write P in its standard form :

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} | Ax = b, x \geq 0 \right\},$$

   with

$$A = \begin{pmatrix} -2 & -2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

   We now investigate the basic solutions. Since we have four variables ($n = 4$) and two constraints ($m = 2$), two variables ($n - m = 2$) need to be out of the basis and equal to 0. Please note that you can also utilize Matlab to invert the matrices.

(a) Basic solution where x_1 and x_2 are in the basis

$$B = \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix};$$

We have $\det(B) = 0$, thus B is not invertible and there is no feasible basic solution. This is a consequence of the parallel constraints, which can be seen in the figure above.

(b) Basic solution where x_1 and x_3 are in the basis

$$B = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}; x = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \end{pmatrix}.$$

The basic solution is feasible and corresponds to point $(2, 0)$ which is denoted by D in the figure.

(c) Basic solution where x_1 and x_4 are in the basis

$$B = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} -1/2 & 0 \\ 1/2 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}; x = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 3/2 \end{pmatrix}.$$

The basic solution is feasible and corresponds to point $(1/2, 0)$ which is denoted by E in the figure.

(d) Basic solution where x_2 and x_3 are in the basis

$$B = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}.$$

The basic solution is feasible and corresponds to point $(0, 2)$ which is denoted by C in the figure.

(e) Basic solution where x_2 and x_4 are in the basis

$$B = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} -1/2 & 0 \\ 1/2 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 3/2 \end{pmatrix}.$$

The basic solution is feasible and corresponds to point $(0, 1/2)$ which is denoted by B in the figure.

(f) Basic solution where x_3 and x_4 are in the basis

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix}.$$

The basic solution, which is denoted by A in the figure, is not feasible since $B^{-1}b \ngeq 0$.

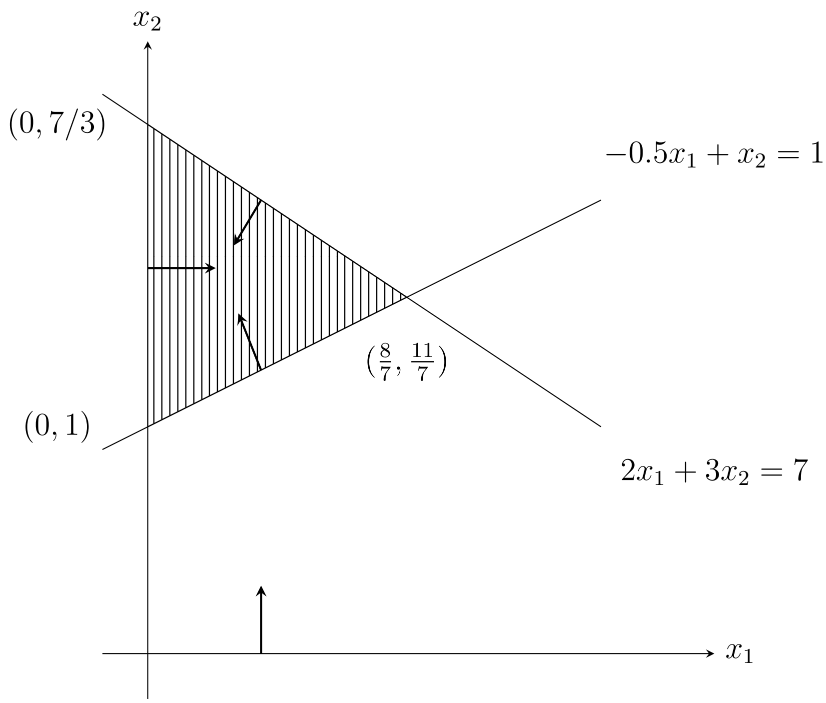
3. We now need only to calculate the values of the objective function :

Point	x_k	f_k
A	$\begin{pmatrix} 0 & 0 & -1 & 2 \end{pmatrix}^T$	$B^{-1}b \ngeq 0$
B	$\begin{pmatrix} 0 & 1/2 & 0 & 3/2 \end{pmatrix}^T$	$1/2$
C	$\begin{pmatrix} 0 & 2 & 3 & 0 \end{pmatrix}^T$	2
D	$\begin{pmatrix} 2 & 0 & 3 & 0 \end{pmatrix}^T$	6
E	$\begin{pmatrix} 1/2 & 0 & 0 & 3/2 \end{pmatrix}^T$	$3/2$

The optimal solution is $x^\star = \begin{pmatrix} 2 & 0 & 3 & 0 \end{pmatrix}^T$. Note that in that exercice, the constraints were parallels and thus could not form a basis.

Solution of the question 2:

The Figure below represents the feasible domain in the space of variables (x_1, x_2).

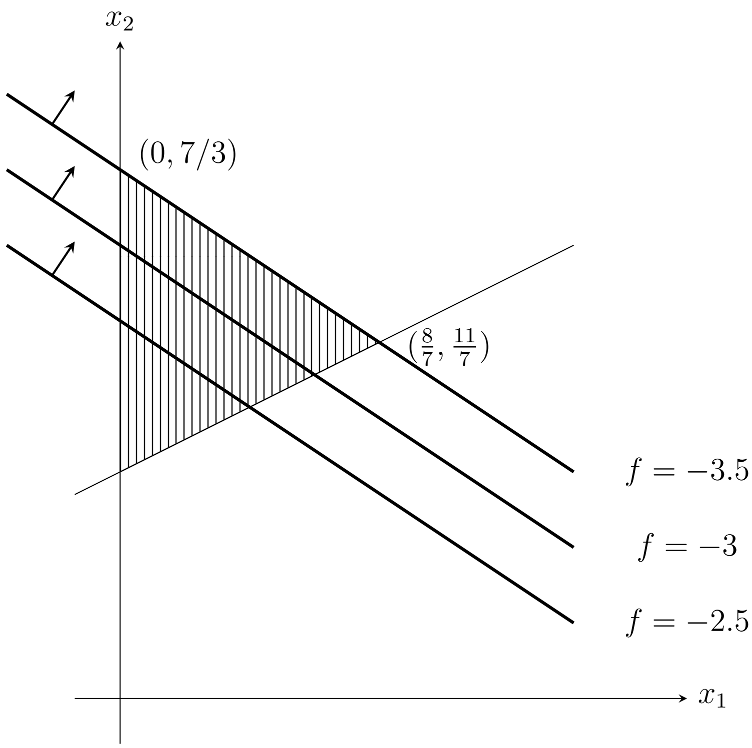


We notice that the constraint $x_1 \geq 0$ can be eliminated. The set of vertices, which corresponds to the intersection of each pair of the remaining constraints, is $\left\{(0,1),(0,\frac{7}{3}),(\frac{8}{7},\frac{11}{7})\right\}$.

We now have to add the level lines corresponding to different values of the objective function. We start by drawing an arbitrary level line intersecting the feasible domain. Then, we calculate the steepest descent direction

$$-\nabla f(x) = -c = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}.$$

That is displayed with an arrow on each level line. Finally, we move this level line parallel to itself as far as possible in the direction of c as long as it intersect the feasible domain.



Here, we observe that the level line is parallel to one border of the polyhedron defined by the constraints. Thus, there is an infinite number of optimal solutions, which correspond to the segment between the vertex $(0, \frac{7}{3})$ and the vertex $(\frac{8}{7}, \frac{11}{7})$, which all have an optimal value of -3.5. In fact, when two vertices have the same optimal value, all the segments between them is optimal.
We deduce that not all optimal solutions are vertices and so they are not all basic solutions.

Solution of the question 3:

1. We consider the set of constraints. In order to write it in standard form, we introduce the slack variables x_3, x_4 and x_5 which lead to the

following system :

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 7 \\ x_1 + x_2 - x_4 &= \frac{3}{2} \\ x_1 - x_5 &= \frac{3}{2} \\ x_1, x_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

We can write it in standard form as the polyhedron

$$\mathcal{P} = \left\{ x \in \mathbb{R}^5 | Ax = b, x \geq 0 \right\},$$

where

$$A = \begin{pmatrix} 1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 3/2 \\ 3/2 \end{pmatrix}.$$

Now we can search the basic solutions. Please note that you can also utilize Matlab to invert the matrices.

— Basic solution with x_1, x_2, x_3 in the basis ($j_1 = 1$, $j_2 = 2, j = 3 = 3$) :

$$B = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -3 & 2 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 3/2 \\ 0 \\ 11/2 \end{pmatrix}.$$

The basic solution

$$x = \begin{pmatrix} 3/2 \\ 0 \\ 11/2 \\ 0 \\ 0 \end{pmatrix}$$

is feasible since all variables are non-negative.

— Basic solution with x_1, x_2, x_4 in the basis ($j_1 = 1$, $j_2 = 2, j_3 = 4$) :

$$B = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1/3 & 0 & -1/3 \\ 1/3 & -1 & 2/3 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 3/2 \\ 11/6 \\ 11/6 \end{pmatrix}.$$

The basic solution

$$x = \begin{pmatrix} 3/2 \\ 11/6 \\ 0 \\ 11/6 \\ 0 \end{pmatrix}$$

is feasible since all variables are non-negative.

— Basic solution with x_1, x_2, x_5 in the basis ($j_1 = 1$, $j_2 = 2, j_3 = 5$) :

$$B = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} ; B^{-1} = \begin{pmatrix} -1/2 & 3/2 & 0 \\ 1/2 & -1/2 & 0 \\ -1/2 & 3/2 & -1 \end{pmatrix} ; x_B = B^{-1}b = \begin{pmatrix} -5/4 \\ 11/4 \\ -11/4 \end{pmatrix}.$$

The basic solution

$$x = \begin{pmatrix} -5/4 \\ 11/4 \\ 0 \\ 0 \\ -11/4 \end{pmatrix}$$

is not feasible since x_1 and x_5 are violating the non-negativity constraints.

— Basic solution with x_1, x_3, x_4 in the basis ($j_1 = 1$, $j_2 = 3, j_3 = 4$) :

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} ; B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} ; x_B = B^{-1}b = \begin{pmatrix} 3/2 \\ 11/2 \\ 0 \end{pmatrix}.$$

The basic solution

$$x = \begin{pmatrix} 3/2 \\ 0 \\ 11/2 \\ 0 \\ 0 \end{pmatrix}$$

is feasible and corresponds to the same vertex as when using $j_1 = 1$, $j_2 = 2, j_3 = 3$ for the basis.

— Basic solution with x_1, x_3, x_5 in the basis ($j_1 = 1,\ j_2 = 3, j_3 = 5$) :

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} ; B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} ; x_B = B^{-1}b = \begin{pmatrix} 3/2 \\ 11/2 \\ 0 \end{pmatrix}.$$

The basic solution

$$x = \begin{pmatrix} 3/2 \\ 0 \\ 11/2 \\ 0 \\ 0 \end{pmatrix}$$

is feasible and corresponds to the same vertex as using $j_1 = 1$, $j_2 = 2, j_3 = 3$ or $j_1 = 1,\ j_2 = 3, j_3 = 4$ for the basis.

— Basic solution with x_1, x_4, x_5 in the basis ($j_1 = 1,\ j_2 = 4, j_3 = 5$) :

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} ; B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} ; x_B = B^{-1}b = \begin{pmatrix} 7 \\ 11/2 \\ 11/2 \end{pmatrix}.$$

The basic solution

$$x = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 11/2 \\ 11/2 \end{pmatrix}$$

is feasible.

— The variables x_2, x_3, x_4 do not form a basis since the corresponding columns of matrix A do not form a full rank matrix.

— Basic solution with x_2, x_3, x_5 in the basis ($j_1 = 2,\ j_2 = 3, j_3 = 5$) :

$$B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; x_B = B^{-1}b = \begin{pmatrix} 3/2 \\ 5/2 \\ -3/2 \end{pmatrix}.$$

The basic solution

$$x = \begin{pmatrix} 0 \\ 3/2 \\ 5/2 \\ 0 \\ -3/2 \end{pmatrix}$$

is not feasible since x_5 is negative.

— Basic solution with x_2, x_4, x_5 in the basis ($j_1 = 2$, $j_2 = 4, j_3 = 5$) :

$$B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; B^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 1/3 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; x_B = B^{-1}b = \begin{pmatrix} 7/3 \\ 5/6 \\ -3/2 \end{pmatrix}.$$

The basic solution

$$x = \begin{pmatrix} 0 \\ 7/3 \\ 0 \\ 5/6 \\ -3/2 \end{pmatrix}$$

is not feasible.

— Basic solution with x_3, x_4, x_5 in the basis ($j_1 = 3$, $j_2 = 4, j_3 = 5$) :

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; x_B = B^{-1}b = \begin{pmatrix} 7 \\ -3/2 \\ -3/2 \end{pmatrix}.$$

The basic solution

$$x = \begin{pmatrix} 0 \\ 0 \\ 7 \\ -3/2 \\ -3/2 \end{pmatrix}$$

is not feasible.

2. We now calculate the reduced costs for each feasible basic solution using the following formula :

$$\bar{c} = c - A^T B^{-T} c_B,$$

where A and B are the matrices of the problem written in standard form, c is the cost vector of the problem, while c_B is the vector composed of the components of the vector c that correspond to the basic variables.

The vector of the costs is obtained from the coefficients in the objective function. In this case, it is

$$c = \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

— First, we consider the basic solution

$$x = \begin{pmatrix} 3/2 \\ 0 \\ 11/2 \\ 0 \\ 0 \end{pmatrix},$$

which corresponds to the vertex $(3/2, 0)$. Its basis is (x_1, x_2, x_3), for which the reduced cost is :

$$
\begin{aligned}
\bar{c} &= \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 3 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 6 \\ 0 \\ -6 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \\ -4 \end{pmatrix}.
\end{aligned}
$$

— Next, we consider the basic solution

$$
x = \begin{pmatrix} 3/2 \\ 11/6 \\ 0 \\ 11/6 \\ 0 \end{pmatrix},
$$

which corresponds to the basic variables (x_1, x_2, x_4), and for which we get

$$
\begin{aligned}
\bar{c} &= \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1/3 & 1/3 \\ 0 & 0 & -1 \\ 1 & -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 1 \\ -1 & 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 6 \\ 2 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

— Further, we consider the basic solution

$$
x = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 11/2 \\ 11/2 \end{pmatrix},
$$

with the basis (x_1, x_4, x_5), for which we get

$$\bar{c} = \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 6 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

Here, we can continue testing the reduced costs of the remaining two basic solutions which correspond to the point $x = (3/2, 0, 11/2, 0, 0)^T$, but we can also observe the following. By evaluating the objective function in each vertex, we observe that $(3/2, 0)$ is optimal with the value of objective function equal to 3.

Since the point $(3/2, 0)$ is degenerate, $\bar{c} \geq 0$ may not hold. However, if there is a descending direction at this point, it must be unfeasible. We will prove this now.
According to the above calculated reduced costs, the only descending

direction at $(3/2, 0)$ corresponds to the basic solution

$$x = \begin{pmatrix} 3/2 \\ 0 \\ 11/2 \\ 0 \\ 0 \end{pmatrix},$$

its basis is (x_1, x_2, x_3), and the non-basic variable x_5. Therefore, the descent direction is calculated with

$$B = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$A_5 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$

and is equal to

$$d_5 = \begin{pmatrix} -B^{-1}A_5 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

We notice that d_5 is not feasible, since the second component is negative and x_2 is equal to 0. This proves that the point $(3/2, 0)$ is optimal, and that it is not required to check the remaining two basic solutions. Even if those were checked, it would yield the same optimal solution since the two remaining basic solutions correspond to the same point $x = (3/2, 0, 11/2, 0, 0)^T$.

Solution of the question 4:

1. The feasible region is represented in Figure 1.

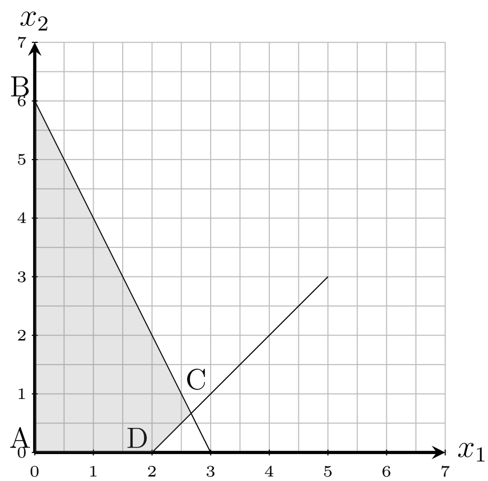


FIGURE 1 – Feasible region

2. To write the problem in standard form, let us introduce two variables, x_3 and x_4 :

$$\min -4x_1 + 3x_2$$

subject to

$$2x_1 + x_2 + x_3 = 6 \quad (1)$$
$$x_1 - x_2 + x_4 = 2 \quad (2)$$
$$x_1, x_2, x_3, x_4 \geq 0$$

3. As specified in the question, first we need to verify that the point (0,0) is a feasible basic solution, in order to start the algorithm with this point. $x_1 = x_2 = 0$ implies that the constraints (1) and (2) become :

$$x_3 = 6,$$
$$x_4 = 2.$$

In other words, the point corresponds to the solution (0, 0, 6, 2). As all the variables are non-negative, we can say that this is a basic feasible solution. This point is denoted by A in Figure 1.

Note : It is also possible to show that $B^{-1}b \geq 0$ holds.
As we now have a starting point, we may start the iterations of the simplex algorithm. Let us identify the matrix A, the vector b and the vector c.

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \text{ and } c = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

First iteration

First, we need to identify the variables that are basic and non-basic.

Basic variables : x_3 and x_4

Non-basic variables : x_1 and x_2

The matrices B and B^{-1} are then given by :

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then x and x_B (x_B corresponds to the vector x with only basic variables and is equal to $B^{-1}b$) become

$$x = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 2 \end{pmatrix} \text{ and } x_B = \begin{pmatrix} 6 \\ 2 \end{pmatrix},$$

It is therefore possible to calculate the objective function value :

$$c_B^T x_B = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 0$$

The objective function we are trying to minimize is currently 0. As the next step of the algorithm, we need to calculate the reduced costs associated with the non-basic variables.

$$\begin{aligned}
\bar{c}_1 &= c_1 - c_B^T B^{-1} A_1 \\
&= -4 - \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
&= -4 - 0 = -4 \\
&\leq 0 \quad \Rightarrow \quad \underline{\text{candidate}}
\end{aligned}$$

$$\begin{aligned}
\bar{c}_2 &= c_2 - c_B^T B^{-1} A_2 \\
&= 3 - \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= 3 - 0 = 3 \\
&\geq 0 \quad \Rightarrow \quad \text{cannot be a candidate}
\end{aligned}$$

The reduced cost is negative only for one of the variables, i.e. x_1. Therefore, it enters the base. (Note that if we had more than one candidate, we would choose the one with the smallest index, according to the Bland's rule.) Now, the basic direction for the next step needs to be calculated. This direction is given by :

$$d_1 = \begin{pmatrix} d_{B_1} \\ d_{N_1} \end{pmatrix}$$

Let us compute the two vectors separately and combine them to obtain d_1 :

$$\begin{aligned}
d_{N_1} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \\
d_{B_1} &= -B^{-1} A_1 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}
\end{aligned}$$

Then the direction is :

$$d = \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix} = d_1.$$

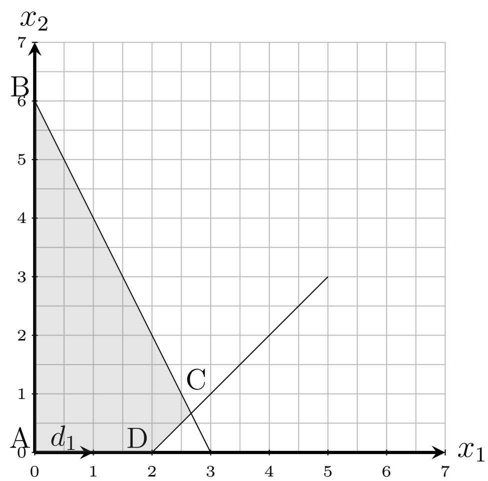


FIGURE 2 – Basic direction corresponding to x_1

This direction is shown in Figure 2.

Now, we need to calculate the distance for the step in the determined direction. For each basic variable, we obtain :

$$\theta_i = \frac{x_i}{-d_i}$$

which gives :

$$\theta_3 = \frac{x_3}{-d_3} = \frac{6}{-(-2)} = 3,$$
$$\theta_4 = \frac{x_4}{-d_4} = \frac{2}{-(-1)} = 2.$$

The maximum step that can be made is then 2. In other words, x_4 leaves the basis and x_1 becomes a basic variable for the next iteration. Moreover, the objective function value decreases by :

$$\theta_4 \times \bar{c}_1 = 2 \times -4 = -8$$

Second iteration

The same procedure as in the first iteration also applies to this iteration. Thus, we first identify the basic and non-basic variables :

Basic variables : x_1 and x_3

Non-basic variables : x_2 and x_4

The matrices B and B^{-1} are then given by :

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

The vectors x and x_B become

$$x = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad x_B = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

which corresponds to point D from Figure 1. In Figure 2, it can be seen that point D is actually the next vertex of feasible polyhedron, going from point A in the direction d_1.

Then the objective function value becomes :

$$c_B^T x_B = \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = -8$$

Notice that this amount corresponds to the decrease that have been calculated at the end of the first iteration. Next, we calculate the reduced costs associated with non-basic variables.

$$\begin{aligned} \bar{c}_2 &= c_2 - c_B^T B^{-1} A_2 \\ &= 3 - \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= 3 - 4 = -1 \\ &\leq 0 \quad \Rightarrow \quad \underline{\text{candidate}} \end{aligned}$$

$$\begin{aligned} \bar{c}_4 &= c_4 - c_B^T B^{-1} A_4 \\ &= 0 - \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= 0 - (-4) = 4 \\ &\geq 0 \quad \Rightarrow \quad \text{cannot be a candidate} \end{aligned}$$

Only one variable is eligible to enter the basis, i.e. x_2. We now calculate the basic direction that the algorithm will take :

$$d_2 = \begin{pmatrix} d_{B_2} \\ d_{N_2} \end{pmatrix}$$

Calculating the two vectors composing the direction to take :

$$d_{N_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} d_2 \\ d_4 \end{pmatrix}$$
$$d_{B_2} = -B^{-1}A_2 = -\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_3 \end{pmatrix}$$

The direction becomes :

$$d = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 0 \end{pmatrix} = d_2.$$

The direction d_2 can be seen in Figure 3.

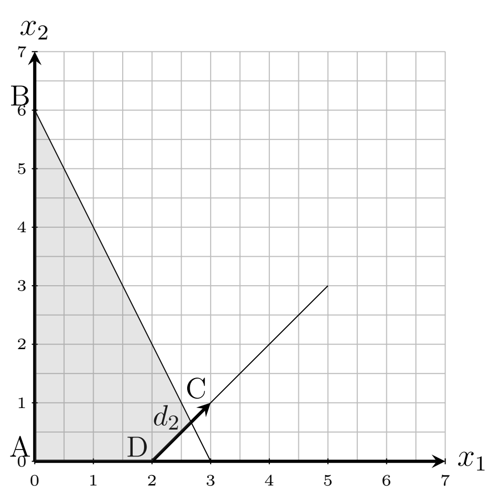


FIGURE 3 – Basic direction corresponding to x_2

We now have to calculate the distance for the step along direction d_2. For each basic variable we obtain :

$$\theta_i = \frac{x_i}{-d_i}$$

which gives :

$$\theta_3 = \frac{x_3}{-d_3} = \frac{2}{-(-3)} = \frac{2}{3},$$

The result implies that the maximum step length is $\frac{2}{3}$ (Note that we cannot have negative steps thus θ_1 is not calculated as $d_1 < 0$.). Then, x_2 will be introduced to the base while x_3 leaves. Also, a decrease in the objective function is expected :

$$\theta_3 \times \bar{c}_2 = \frac{2}{3} \times -1 = -\frac{2}{3}$$

Third iteration

Let us identify the basic and non-basic variables :

Basic variables : x_1 and x_2

Non-basic variables : x_3 and x_4

The matrices B and B^{-1} are then given by :

$$B = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \quad \Rightarrow \quad B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

Then vectors x and x_B are

$$x = \begin{pmatrix} 8/3 \\ 2/3 \\ 0 \\ 0 \end{pmatrix} \text{ and } x_B = \begin{pmatrix} 8/3 \\ 2/3 \end{pmatrix},$$

which is the point C. Then we can calculate the objective function value as follows :

$$c_B^T x_B = \begin{pmatrix} -4 & 3 \end{pmatrix} \begin{pmatrix} 8/3 \\ 2/3 \end{pmatrix} = -\frac{26}{3}$$

The objective function we are trying to minimize is now $-\frac{26}{3}$. This corresponds to the decrease that has been calculated at the end of previous iteration. Now we need to calculate the reduced costs associated to the non-basic variables.

$$\begin{aligned}
\bar{c}_3 &= c_3 - c_B^T B^{-1} A_3 \\
&= 0 - \begin{pmatrix} -4 & 3 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= 0 - \frac{1}{3}(-1) = \frac{1}{3} \\
&\geq 0 \quad \Rightarrow \quad \text{cannot be a candidate}
\end{aligned}$$

$$\begin{aligned}
\bar{c}_4 &= c_4 - c_B^T B^{-1} A_4 \\
&= 0 - \begin{pmatrix} -4 & 3 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= 0 - \frac{1}{3}(-10) = \frac{10}{3} \\
&\geq 0 \quad \Rightarrow \quad \text{cannot be a candidate}
\end{aligned}$$

Since the reduced costs of the two non-basic variables are positive, it can be concluded that the optimum has been reached. Then the solution to the problem

$$\min -4x_1 + 3x_2$$

subject to

$$\begin{aligned}
2x_1 + x_2 + x_3 &= 6 \qquad (3) \\
x_1 - x_2 + x_4 &= 2 \qquad (4) \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}$$

is given by

$$x^* = \begin{pmatrix} 8/3 \\ 2/3 \\ 0 \\ 0 \end{pmatrix}$$

and the objective function value is $-\frac{26}{3}$. Figure 4 shows the graphical solution of the problem, which is also the same with the result that we obtained from algebraic solution.

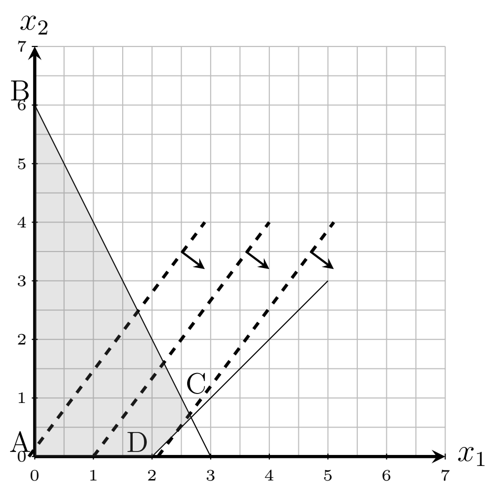


FIGURE 4 – Graphical solution

Solution of the question 5 – Multiple Choice Questions:

1. The correct answer is (c).

   We can easily deduce that (a) is wrong as the main idea behind the simplex method is to preserve feasibility while changing basis. For instance, after determining the basic direction, we need to identify the step length in order not to end up out of the feasible region. (b) is also not correct, as nonbasic variables may also have zero reduced cost. (c) is correct as this is also the stopping condition of the simplex method.
2. The correct answer is (a).

   (c) is definitely not correct as we largen the feasible region by increasing the right hand side of an inferior constraint. As the feasible region gets larger the objective function value cannot increase, it either stays the same or decreases (If it was a maximization problem, it would either stay the same or increase). The statement in (b) is also not correct. We may have an unbounded feasible region but we do not know whether the objective function is in the direction of unboundedness. (a) is correct and an example can also be seen in the 2^{nd} question.