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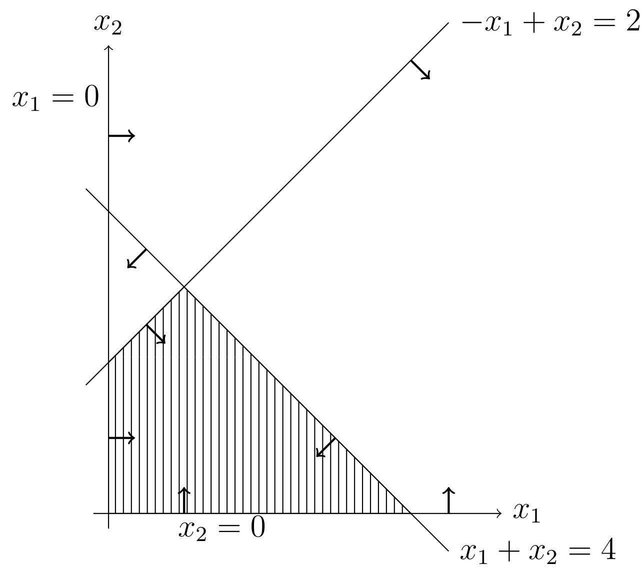
Introduction à l'optimisation linéaire–corrigé (28 September 2018)

Solution of the question 1:

1. Each of the four constraints defines a half-plane, denoted with a pair of arrows. The arrows for the constraint j correspond to the vector $-a_j$, i.e. to the negation of j^{th} row of matrix A. The intersection of the half-planes gives the polyhedron in the $\mathbb{R}^2$ space.



2. In order to write the standard form of the polyhedron, we need to include the slack variables in the following way :

$$\mathcal{P} = \{x \in \mathbb{R}^2, x^s \in \mathbb{R}^4 \mid Ax + x^s = b, x \geq 0, x^s \geq 0\}.$$

Therefore, the problem in standard form is :

$$\begin{aligned} -x_1 + x_2 + x_3 &= 2, \\ x_1 + x_2 + x_4 &= 4, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Slack variables represent the distance between the point and the constraint corresponding to the slack variable. Given a problem in the standard form and its solution, the active equality constraints are those for which the corresponding slack variable is equal to 0, while the active non-negativity constrains are those where the variable is equal to 0.

Solution of the question 2:

1. Each constraint corresponds to a half-plain bounded by either one of the axes or one of the lines denoted with (1)-(4). Our polyhedron represents an intersection of six such half-plains.

   The constraints corresponding to the axes have the form : $x_i \, \theta \, 0$, $i \in \{1, 2\}$, $\theta \in \{\leq, \geq\}$. From the Figure 1 in the question, we see that the polyhedron belongs to the positive half-plains of both axes and therefore, we have the following constraints :

   $$x_1 \geq 0,$$
   $$x_2 \geq 0.$$

   The constraints corresponding to the lines (1)-(4) can be expressed in the following form : $ax_1 + x_2 \, \theta \, b$, $\theta \in \{\leq, \geq\}$. To determine the constraint, we first need to determine the line equation in the form $ax_1 + x_2 = b$, what can be done by reading the two line points from the graph, and solving the system of two equations with two unknowns. The constraint inequation is further obtained by selecting any point (x_1^*, x_2^*) from the polyhedron and choosing $\theta \in \{\leq, \geq\}$ such that $ax_1^* + x_2^* \, \theta \, b$ holds.

   For example, for line (1), we can read from the graph that points $(x_1, x_2) = (0, 2)$ and $(x_1, x_2) = (2, 4)$ belong to it, which further yields the following system of equations :

   $$2 = b,$$
   $$2a + 4 = b,$$

   with the solution $(a, b) = (-1, 2)$ and the line equation $-x_1 + x_2 = 2$. Again, from the graph we can see that we should choose the half-plain which the point $(1, 1)$ belongs to. In this case, it is the half-plain $-x_1 + x_2 \leq 2$, which is also the first identified constraint.

   If we apply the same principle to all lines, we obtain the following list of all constraints :

$$
\begin{aligned}
-x_1 + x_2 &\leq 2,\\
2x_1 + x_2 &\leq 8,\\
x_1 + x_2 &\geq 1,\\
x_1 + 2x_2 &\leq 10,\\
x_1 &\geq 0,\\
x_2 &\geq 0.
\end{aligned}
$$

2. The optimization problem, given in the standard form, is written as :

$$
\begin{array}{lrrrrrrcll}
\min & -x_1 & +2x_2 & & & & & & & \\
\text{s.t.} & -x_1 & +x_2 & +x_3 & & & & = & 2 & \\
 & 2x_1 & +x_2 & & +x_4 & & & = & 8 & \\
 & x_1 & +x_2 & & & -x_5 & & = & 1 & \\
 & x_1 & +2x_2 & & & & +x_6 & = & 10 & \\
 & & & & & & x_i & \geq & 0 & \forall i \in 1, \ldots, 6.
\end{array}
$$

In the given problem, we have $m = 4$ constraints and $n = 6$ variables. The matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and the vectors $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$ are equal to :

$$
\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 2 \\ 8 \\ 1 \\ 10 \end{pmatrix}, \qquad \boldsymbol{c} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
$$

3. The following table presents all basic solutions and denotes them as feasible or not. Each basic solution is obtained by setting $n - m = 2$ variables to be non-basic ones, i.e. to be equal to 0, and by calculating other variable values using the equality constraints from the standard form $x_B = \boldsymbol{B}^{-1}\boldsymbol{b}$. Consequently, a basic solution exists only if the matrix $\boldsymbol{B}$ is not singular. The feasible basic solutions are the the ones which do not violate the non-negativity constrains.

Non-basic var.	x_1	x_2	x_3	x_4	x_5	x_6	Feasible ?
x_5, x_6	-8	9	-15	15	0	0	no
x_4, x_6	2	4	0	0	5	0	yes
x_3, x_6	2	4	0	0	5	0	yes
x_2, x_6	10	0	12	-12	9	0	no
x_1, x_6	0	5	-3	3	2	0	no
x_4, x_5	7	-6	15	0	0	15	no
x_3, x_5	-1/2	3/2	0	15/2	0	15/2	no
x_2, x_5	1	0	3	6	0	9	yes
x_1, x_5	0	1	1	7	0	8	yes
x_3, x_4	2	4	0	0	5	0	yes
x_2, x_4	4	0	6	0	3	6	yes
x_1, x_4	0	8	-6	0	7	-6	no
x_2, x_3	-2	0	0	12	-3	12	no
x_1, x_3	0	2	0	6	5	6	yes
x_1, x_2	0	0	2	8	-1	10	no

From the table above, it can be inferred that the feasible basic solutions correspond to the following vertices of the constraint polyhedron :

A(0,2), B(0,1), C(1,0), D(4,0) and E(2,4).

With the same letters, the feasible basic solutions are denoted in Figure 1.

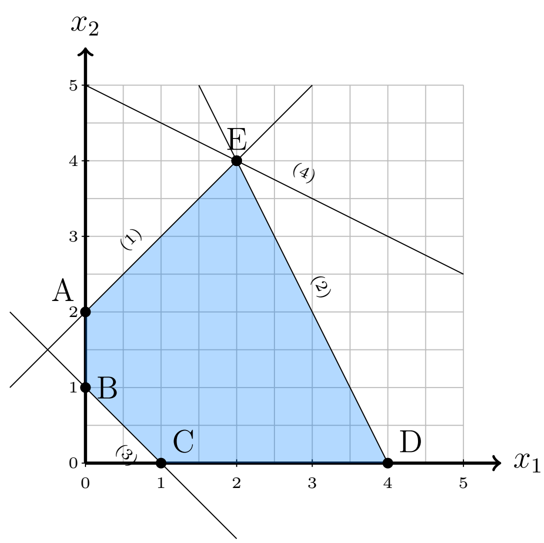


Figure 1 – Basic solutions

The basic solutions correspond to the intersections of the constraints. The feasible basic solutions with non-basic variables (x_3, x_4), (x_3, x_6), and (x_4, x_6) are in fact a single degenerate feasible basic solution, since more than $n - m = 2$ variables are equal to 0, i.e. more than two constraints are active in this point. In Figure 1, this degenerate solution is denoted with E, and it can be observed that three constraints ((1), (2), and (4)) are reached at this point.

4. For the given problem in standard form, the point $(3, 2)$ corresponds to the point $(x_1, x_2, x_3, x_4, x_5, x_6) = (3, 2, 3, 0, 4, 2)$. It is a feasible solution since it satisfies all constraints. However, in order for it to be a basic solution, at least $n - m = 2$ variables have to be equal to 0, which is not the case.

Solution of the question 3:

First, the polyhedron needs to be written in standard form. In order to do so, we need to transform the constraints in equality constraints by introducing the slack variables $x_1^s = x_3$ and $x_2^s = x_4$ for the first and second equation respectively. The constraints are then written as follows :

$$
\begin{aligned}
-x_1 + x_2 + x_3 \qquad &= 2,\\
x_1 + x_2 \qquad + x_4 &= 4,\\
x_1, x_2, x_3, x_4 &\geq 0.
\end{aligned}
$$

Given a feasible point of the polyhedron, it is possible to analyze if a direction is feasible by checking the following two conditions :

- $Ad = 0$, and
- $d_i \geq 0$ when $x_i = 0$,

where

$$
A \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}.
$$

Let's verify if these conditions are satisfied for the four cases.

1. $x_a = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $d_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

   We first need to compute the values of the slack variables x_3 and x_4 and we obtain $x_3 = 0$ and $x_4 = 2$. In order to satisfy the first condition, we must have

$$
Ad = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} -1 + d_3 \\ 1 + d_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

   from which we obtain $d_3 = 1$, $d_4 = -1$, and $d = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$.

   This satisfies the first condition. We finally observe that the second condition is also satisfied, since $d_1 = 1 \geq 0$ when $x_1 = 0$ and $d_3 = 1 \geq 0$ when $x_3 = 0$. We conclude that the given direction is feasible for the given point.

2. $x_b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $d_b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

In this case it is immediate to see that the direction is not feasible, since $d_1 = -1 \leq 0$ when $x_1 = 0$.

3. $x_c = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $d_c = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

   We compute the values of the slack variables : $x_3 = 2$, $x_4 = 0$. For the first condition to be satisfied, we must have

$$Ad = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} 2 + d_3 \\ 2 + d_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

   from which we obtain $d_3 = -2$, $d_4 = -2$, and finally, $d = \begin{pmatrix} 0 \\ 2 \\ -2 \\ -2 \end{pmatrix}$.

   We now check the second condition and we observe that $d_4 = -2 \leq 0$ when $x_4 = 0$. The second condition is not satisfied, therefore the given direction is not feasible for the given point.

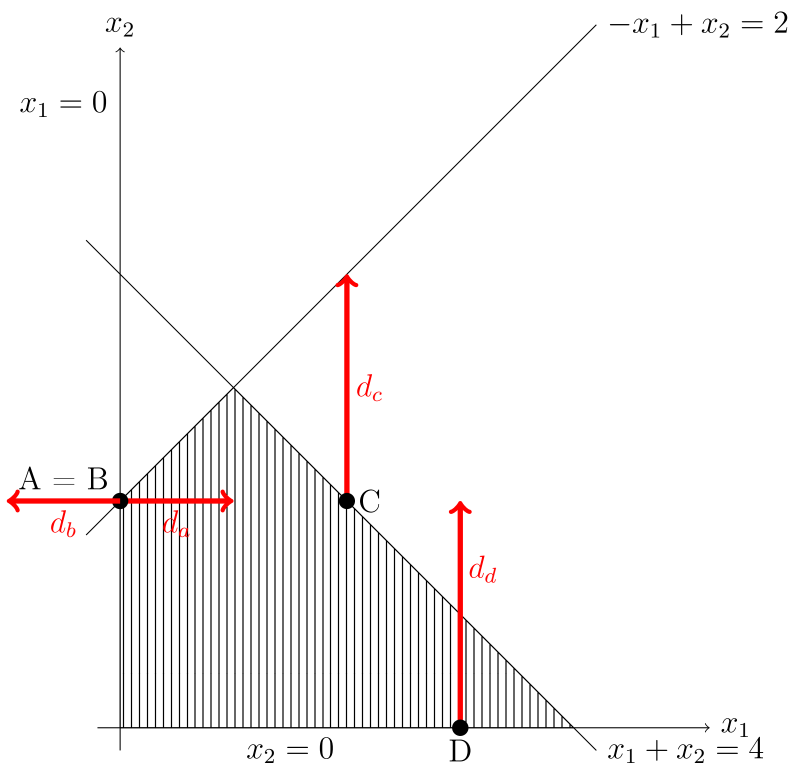
4. $x_d = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $d_d = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

   We compute the values of the slack variables : $x_3 = 5$, $x_4 = 1$. For the first condition to be satisfied, we must have

$$Ad = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} 2 + d_3 \\ 2 + d_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

   from which we obtain $d_3 = -2$ and $d_4 = -2$, and $d = \begin{pmatrix} 0 \\ 2 \\ -2 \\ -2 \end{pmatrix}$, as in the previous case. We now check the second condition and we observe that $d_2 = 2 \geq 0$ when $x_2 = 0$. The second condition is therefore satisfied. This means that the given direction is feasible for the given point.

The following figure presents the four studied cases.



Solution de la question 4 – QCM:

1. (d)

   En effet, dans (a), la première contrainte représente le fait que le café Super et le café Deluxe contiennent chacun 50% de café brésilien, alors que la seconde contrainte représente le fait que le café Super et le café Deluxe contiennent respectivement 30% et 70% de café colombien. Ceci ne correspond pas à la description du problème. De plus, ces contraintes sont contradictoires puisqu'elles permettent de spécifier 120% des ingrédients du café Deluxe.

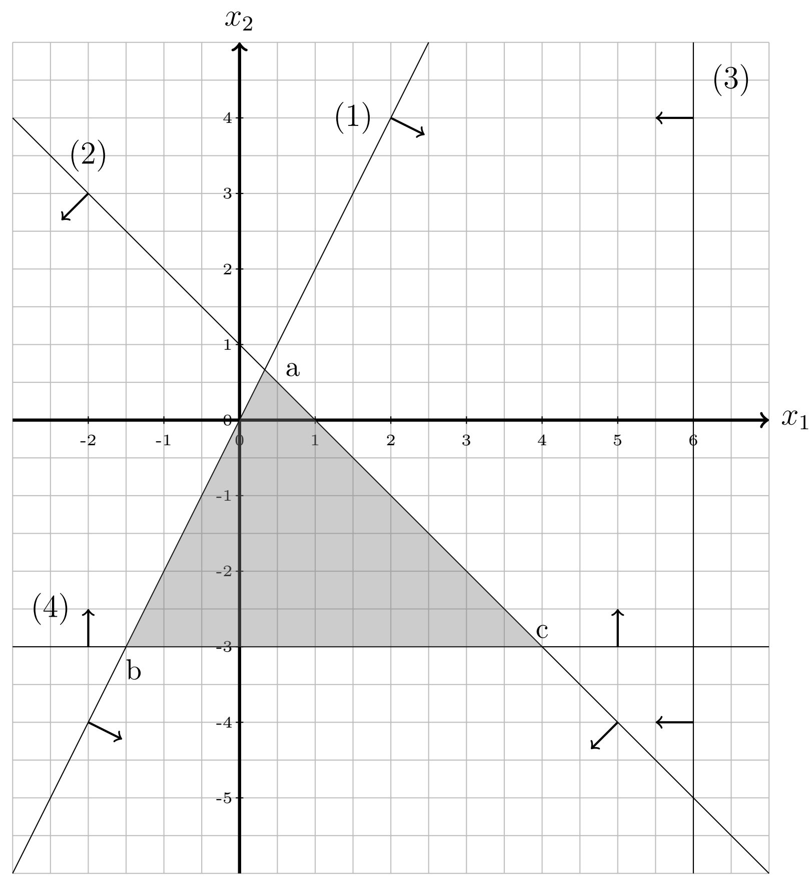
   Dans (b), la fonction objectif modélise un bénéfice de 1 CHF pour chaque kilogramme des deux types de café. La première contrainte représente le fait que le café Super est constitué à 80% de café brésilien. La deuxième contrainte représente le fait que le café Deluxe est constitué à 120% de café colombien. Tout ceci est contradiction avec les données du problème.

   Dans (c), la fonction objectif ainsi que les contraintes sont corrects. Par contre, il manque les contraintes d'integrité (la non négativité des variables).

   Finalement, (d) correspond exactement à la description du problème.

2. (b) Incompatible signifie qu'aucune solution ne vérifie les contraintes. La réponse est donc 0, c'est-à-dire (b).

3. (d) Le domaine admissible est donné ci-dessous :



Les sommets sont a=(1/3, 2/3), b=(-1.5, -3) et c=(4, -3). De plus, on voit que la contrainte (3) est redondante.

4. (c) Comme pour l'exercice 2, nous devons d'abord écrire le problème sous forme standard :

$$
\begin{aligned}
\min\ & -x_1 + 4x_2 + 3x_3 \\
\text{s.c.}\ & -x_1 + 4x_2 + x_4 = 2 \\
& 2x_1 - x_3 + x_5 = 2 \\
& x_2 + 2x_3 - x_6 = 5 \\
& x_i \geq 0, \quad \forall i \in 1, \ldots, 6.
\end{aligned}
$$

Notons qu'il y a $n = 6$ variables et $m = 3$ contraintes pour ce problème.

Pour chacun des points, nous devons calculer la valeur des variables d'écart. Pour cela, nous utilisons les contraintes d'égalité du problème. Nous vérifions si le point est admissible c'est-à-dire si toutes les variables de décision sont positives. Enfin, nous verifions si le point satisfait les conditions de la definition 3.38 (solution de base).

(a) $(x_1, x_2, x_3, x_4, x_5, x_6) = (2, 0, 2, 4, 0, -6)$
Ce point ne correspond pas à une solution admissible car $x_6 \leq 0$. De plus, il ne s'agit pas d'une solution de base puisque le point ne vérifie pas la deuxième condition de la définition 3.38 qui stipule qu'au moins $n - m = 3$ variables de decision doivent être égale à 0.

(b) $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, -1, 1, -2)$
Ce point ne correspond pas à une solution admissible car $x_4 \leq 0$ et $x_6 \leq 0$. De plus, il ne s'agit pas d'une solution de base puisque le point ne vérifie pas la deuxième condition de la définition 3.38 qui stipule qu'au moins $n - m = 3$ variables de decision doivent être égale à 0.

(c) $(x_1, x_2, x_3, x_4, x_5, x_6) = (2, 1, 2, 0, 0, 0)$
La solution est admissible puisque toutes les variables de decisions sont positives. De plus la solution satisfait la deuxième condition de la définition 3.38. En effet, $n - m = 3$ variables sont nulles. Il ne reste plus qu'à vérifier que la matrice $\boldsymbol{B}$ est inversible ce qui est le cas ssi $\det \boldsymbol{B} \neq 0$. Dans ce cas,

$$\boldsymbol{B} = \begin{pmatrix} -1 & 4 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

et

$$\det \boldsymbol{B} = b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{23}b_{32}b_{11} - b_{33}b_{12}b_{21} = 0 + 0 + 0 - 0 - 1 - 16 = -17.$$

On peut ainsi déduire que ce point correspond à une solution admissible de base.

(d) $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 2, 2, -5, 2, 1)$
Ce point ne correspond pas à une solution admissible car $x_4 \leq 0$. De plus, il ne s'agit pas d'une solution de base puisque le point ne

vérifie pas la deuxième condition de la définition 3.38 qui stipule qu'au moins $n - m = 3$ variables de decision doivent être égale à 0.