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Question 1:

For each of the following linear optimization problems, write the dual problem by using the summary table of Theorem 4.14, i.e. without explicitly writing the Lagrangian function.

1.
$$\min x_1 - 4x_2 + 2x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 10, \\ x_1 + 4x_3 &= 16, \\ x_2 - x_3 &\leq 5, \\ x_1 &\geq 0, \\ x_2 &\in \mathbb{R}, \\ x_3 &\leq 0. \end{aligned}$$

2.
$$\min -4x_1 - 7x_2$$

subject to

$$\begin{aligned} -5x_1 + 4x_2 &\leq 16, \\ x_1 + 3x_2 &\leq 31, \\ x_1 + 2x_2 &\leq 24, \\ 3x_1 + 5x_2 &\leq 68, \\ x_1 + x_2 &\leq 122, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

Question 2:

For the following optimization problem, where $n = 2m$, $m \in \mathbb{N}$, write the dual problem.

$$\min x_1 + 2x_2 + \ldots + nx_n$$

subject to

$$\begin{aligned} x_1 + x_2 &\geq 2 \\ x_1 + x_2 + x_3 + x_4 &\geq 4 \\ &\vdots \\ x_1 + x_2 + x_3 + ... + x_n &\geq n \\ x_1, x_2, x_3, ..., x_n &\geq 0 \end{aligned}$$

Question 3:

Soit la matrice carrée $A \in \mathbb{R}^{n \times n}$ telle que $A^T = -A$, et le vector $c \in \mathbb{R}^n$. Prouver que le problème d'optimisation linéaire suivant :

$$\min c^T x$$

sous contraintes

$$\begin{aligned} Ax &\geq -c, \\ x &\geq 0, \end{aligned}$$

est auto-dual, c'est-à-dire qu'il est son propre problème dual.

Question 4:

For the following optimization problem, write the Lagrangian function and the dual problem. Then, discuss the feasibility of the primal and dual problems.

$$\min_{x \in \mathbb{R}^2} 4x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\geq 2 \\ -x_1 - x_2 &\geq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$

Question 5:

Consider the following linear problem :

$$\max 2x_1 + 7x_2 + 3x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 - x_3 &\geq 1 \\ -x_1 - 2x_2 &\geq 2 \\ x_1, x_2, x_3 &\leq 0. \end{aligned}$$

1. Write the dual problem.
2. Solve the dual problem by the graphical method.
3. Using the *complementarity slackness* conditions, determine the optimal solution to the primal problem.
4. Verify that the *strong duality* theorem holds.
5. Verify *corollary 6.32*, which claims that the optimal solution to the dual problem is given by

   $$B^{-T} c_B,$$

   where B is the basis matrix of the primal problem.