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Dualité - corrigé (16 novembre 2018)

## Solution of the question 1:

1. Using the summary table of Theorem 4.14, we get that the dual problem is :

$$\max 10y_1 + 16y_2 + 5y_3$$

subject to

$$\begin{aligned} 2y_1 + y_2 &\leq 1, \\ y_1 + y_3 &= -4, \\ 4y_2 - y_3 &\geq 2, \\ y_1 &\geq 0, \\ y_2 &\in \mathbb{R}, \\ y_3 &\leq 0. \end{aligned}$$

2. Using the summary table of Theorem 4.14, we get that the dual problem is :

$$\max 16y_1 + 31y_2 + 24y_3 + 68y_4 + 122y_5$$

subject to

$$\begin{aligned} -5y_1 + y_2 + y_3 + 3y_4 + y_5 &\leq -4, \\ 4y_1 + 3y_2 + 2y_3 + 5y_4 + y_5 &\leq -7, \\ y_1, y_2, y_3, y_4, y_5 &\leq 0. \end{aligned}$$

## Solution of the question 2:

Using the summary table of Theorem 4.14, we obtain the following dual problem :

$$\max 2y_1 + 4y_2 + ... + 2my_m$$

subject to

$$
\begin{array}{cccccccr}
y_1 & +y_2 & +y_3 & +y_4 & +... & +y_m & & \leq 1, \\
y_1 & +y_2 & +y_3 & +y_4 & +... & +y_m & & \leq 2, \\
 & y_2 & +y_3 & +y_4 & +... & +y_m & & \leq 3, \\
 & y_2 & +y_3 & +y_4 & +... & +y_m & & \leq 4, \\
 & & & & ... & ...... & & ... \\
 & & & & & y_m & & \leq n-1, \\
 & & & & & y_m & & \leq n, \\
y_1, & y_2, & y_3, & y_4, & ... & y_m & & \geq 0.
\end{array}
$$

In the above problem, we see that every second constraint is redundant, and can be omitted. Thus, we obtain the following problem :

$$\max 2y_1 + 4y_2 + ... + 2my_m$$

subject to

$$
\begin{array}{cccccccr}
y_1 & +y_2 & +y_3 & +y_4 & +... & +y_m & & \leq 1, \\
 & y_2 & +y_3 & +y_4 & +... & +y_m & & \leq 3, \\
 & & & & ... & ...... & & ... \\
 & & & & & y_m & & \leq n-1, \\
y_1, & y_2, & y_3, & y_4, & ... & y_m & & \geq 0.
\end{array}
$$

Finally, since $y_i \geq 0$, $i \in \{1, ..., m\}$, all constraints except for the first one and the non-negativity constraints are redundant, and we obtain the following problem :

$$\max 2y_1 + 4y_2 + ... + 2my_m$$

subject to

$$
\begin{array}{ccccccr}
y_1 & +y_2 & +y_3 & +y_4 & +... & +y_m & \leq 1, \\
y_1, & y_2, & y_3, & y_4, & ... & y_m & \geq 0.
\end{array}
$$

Here it should be mentioned that this is an example where the dual problem is simpler than the primal since it contains less variables and less constraints.

## Solution de la question 3:

Le problème dual s'écrit :

$$\max -c^T y$$

sous contraintes

$$\begin{aligned} A^T y &\leq c \\ y &\geq 0. \end{aligned}$$

On peut réécrire le problème en utilisant le fait que

1. $\max -f(y) \Leftrightarrow \min f(y)$,
2. $A^T = -A$.

On obtient :

$$\min c^T y$$

sous contraintes

$$\begin{aligned} -Ay &\leq c \\ y &\geq 0. \end{aligned}$$

Finalement, en multipliant la contrainte d'inégalité par -1 et en renommant simplement la variable $y$ par $x$, on obtient :

$$\min c^T x$$

sous contraintes

$$\begin{aligned} Ax &\geq -c \\ x &\geq 0, \end{aligned}$$

ce qui correspond bien au problème primal de départ.

**Solution of the question 4:**

In the given optimization problem, if the two first constraints are observed, we can deduce that the primal problem is not feasible. Thus, the dual is either infeasible or unbounded. We will show this by deriving the dual problem.

We can write the optimization problem as

$$\min_{x \in \mathbb{R}^2} 4x_1 + 2x_2$$

subject to

$$\begin{aligned} g_1(x) &= 2 - x_1 - x_2 \leq 0 & (\mu_1) \\ g_2(x) &= 1 + x_1 + x_2 \leq 0 & (\mu_2) \\ g_3(x) &= \quad - x_1 \qquad \leq 0 & (\mu_3) \\ g_4(x) &= \qquad\quad - x_2 \leq 0 & (\mu_4). \end{aligned}$$

The Lagrangian function of this problem is

$$\begin{aligned} L(x_1, x_2, \mu_1, \mu_2, \mu_3, \mu_4) &= 4x_1 + 2x_2 + \mu_1(2 - x_1 - x_2) \\ &\quad + \mu_2(1 + x_1 + x_2) - \mu_3 x_1 - \mu_4 x_2 \\ &= (4 - \mu_1 + \mu_2 - \mu_3)x_1 \\ &\quad + (2 - \mu_1 + \mu_2 - \mu_4)x_2 + 2\mu_1 + \mu_2. \end{aligned}$$

In order for the dual function to be bounded, the coefficients of $x_1$ and $x_2$ have to be zero, i.e. :

$$\begin{aligned} \mu_1 - \mu_2 + \mu_3 - 4 = 0, &\text{ and} \\ \mu_1 - \mu_2 + \mu_4 - 2 = 0. & \end{aligned}$$

Therefore, we obtain the following dual problem :

$$\max 2\mu_1 + \mu_2$$

subject to

$$\begin{aligned} \mu_1 - \mu_2 + \mu_3 &= 4, \\ \mu_1 - \mu_2 + \mu_4 &= 2, \\ \mu_1 &\geq 0, \\ \mu_2 &\geq 0, \\ \mu_3 &\geq 0, \\ \mu_4 &\geq 0. \end{aligned}$$

Further, we can observe that $\mu_3$ and $\mu_4$ act as slack variables and simplify the problem by removing them, to obtain the following formulation :

$$\max 2\mu_1 + \mu_2$$

subject to

$$\begin{aligned} \mu_1 - \mu_2 &\leq 4, \\ \mu_1 - \mu_2 &\leq 2, \\ \mu_1 &\geq 0, \\ \mu_2 &\geq 0. \end{aligned}$$

At this point, we can easily see that the constraint $\mu_1 - \mu_2 \leq 4$ is redundant since it can be inferred from the next constraint $\mu_1 - \mu_2 \leq 2$. Thus, the former can be removed.

Therefore, the dual function domain is

$$X_q = \{\mu_1, \mu_2, \mu_3, \mu_4 | \mu_1 \leq \mu_2 + 2, \mu_1 \geq 0, \mu_2 \geq 0\},$$

and the final version of the dual problem can be written as

$$\max 2\mu_1 + \mu_2$$

subject to

$$\begin{aligned} \mu_1 - \mu_2 &\leq 2, \\ \mu_1 &\geq 0, \\ \mu_2 &\geq 0, \end{aligned}$$

which is feasible but unbounded. It comes from the fact that there is an upper bound only for the difference between $\mu_1$ and $\mu_2$, but not for those variable themselves.

Also, please note that the unboundedness of the dual problem is expected when the primal problem is infeasible, according to the Theorem 4.9 (Weak Duality) and Corollary 4.12.

## Solution of the question 5:

1. The dual problem is

$$\min \quad y_1 + 2y_2$$

subject to

$$\begin{aligned} 2y_1 - y_2 &\leq 2, \\ y_1 - 2y_2 &\leq 7, \\ -y_1 &\leq 3, \\ y_1, y_2 &\leq 0. \end{aligned}$$

2. The graphical solution to the dual problem is given in Figure **??**. The optimal solution to the dual denoted by point $E$ is

$$\begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \end{pmatrix},$$

and the optimal objective function value is $-13$.

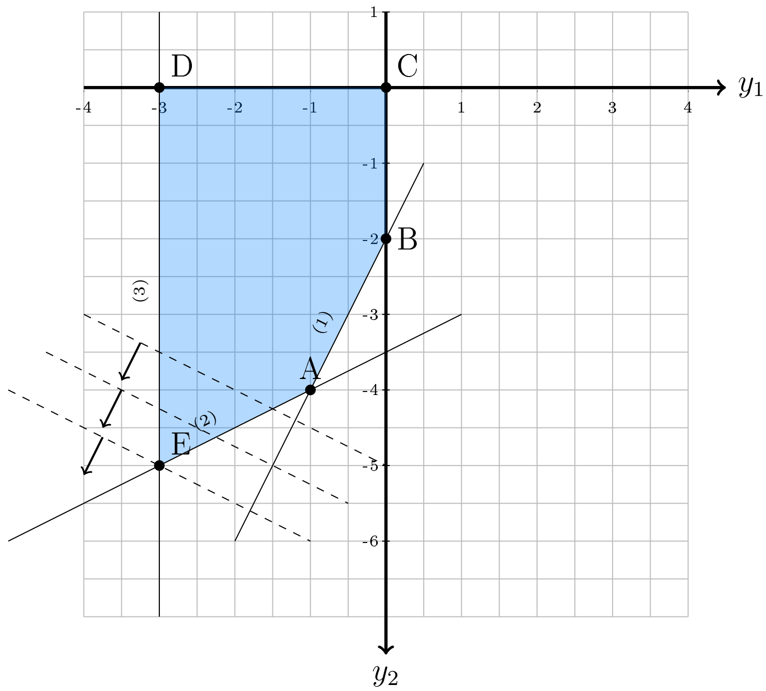


Figure 1 – Graphical solution to the dual

3. Theorem 6.34 states the complementarity slackness as follows.
   Consider the primal linear problem

$$min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$Ax = b$$
$$x \geq 0,$$

where $A \in \mathbb{R}^{mxn}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, and the dual problem

$$max_{\lambda \in \mathbb{R}^m} \lambda^T b$$

subject to

$$A^T\lambda \leq c.$$

Consider $x^*$ primal feasible and $\lambda^*$ dual feasible. $x^*$ is optimal for the primal and $\lambda^*$ is optimal for the dual if and only if

$$(c_i - \sum_{j=1}^{m} a_{ji}\lambda_i)x_i = 0, \quad i = 1, ..., n.$$

In order to utilize this theorem, we first write the standard form of the original problem. We change the variables $x_1$, $x_2$, and $x_3$ and include $x'_1$, $x'_2$, and $x'_3$ to have nonnegative decision variables.

$$x'_1 = -x_1, \quad x'_2 = -x_2, \quad x'_3 = -x_3,$$

Then the standard form becomes

$$\min \quad 2x'_1 + 7x'_2 + 3x'_3$$

subject to

$$\begin{aligned} -2x'_1 - x'_2 + x'_3 - e_1 &= 1 \\ x'_1 + 2x'_2 - e_2 &= 2 \\ x'_1, x'_2, x'_3 &\geq 0. \end{aligned}$$

The dual of the standard form is then

$$\max \quad y_1 + 2y_2$$

subject to

$$\begin{aligned} -2y_1 + y_2 &\leq 2, \\ -y_1 + 2y_2 &\leq 7, \\ y_1 &\leq 3, \\ y_1, y_2 &\in \mathbb{R}. \end{aligned}$$

Please notice that as we made change of variables, the dual problem above is not the same with the dual we have written in the first part.

Now, we solve the dual problem by graphical method and find the optimal solution. From the graph, we see that the solution is

$$\begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix},$$

and the optimal objective function value is 13.

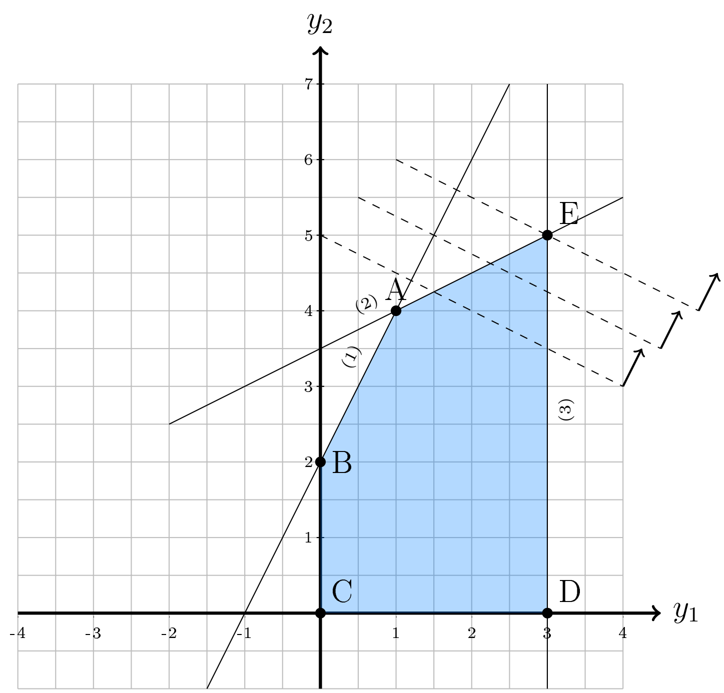


FIGURE 2 – Graphical solution to the dual

We then write the conditions that need to be satisfied :

$$(c_1 - (-2y_1 + y_2))x_1' = 0$$
$$(c_2 - (-y_1 + 2y_2))x_2' = 0$$
$$(c_3 - (y_1))x_3' = 0$$

where $c^T = (2, 7, 3)$ and $y_1^* = 3$, $y_2^* = 5$.

$$3x_1' = 0$$

$$0x'_2 = 0$$
$$0x'_3 = 0$$

To satisfy these equations, we need to have $x'_1 = 0$ and we cannot deduce any information for $x'_2$ and $x'_3$. On the other hand, we know that $y_1$ and $y_2$ are nonzero which means that the constraints in the primal problem should be active, i.e. $e_1 = e_2 = 0$. We evaluate the primal constraints by putting 0 in the place of $x'_1$ and solving the system of equations below

$$-x'_2 + x'_3 = 1$$
$$2x'_2 = 2$$

we obtain the optimal solution to the primal problem as $(x'^*_1, x'^*_2, x'^*_3) = (0, 1, 2)$. If we change back to the original variables, we obtain $(x^*_1, x^*_2, x^*_3) = (0, -1, -2)$ and the objective function value becomes $-13$.

4. The theorem states that if either the primal or the dual problem has an optimal solution, so does the other, and the optimal objective values are equal.

   The previous part already shows that both problems have an optimal solution. The objective function value to the primal is $2*0+7*(-1)+3*(-2) = -13$ and to the dual $(-3)+2*(-5) = -13$ which are equal.

5. The primal problem has $x_2$ and $x_3$ in the basis at the optimality. Thus, the optimal basis matrix is given by

$$B = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$$

and

$$B^{-1} = \begin{pmatrix} 0 & -1/2 \\ -1 & -1/2 \end{pmatrix}$$

and

$$B^{-T} = \begin{pmatrix} 0 & -1 \\ -1/2 & -1/2 \end{pmatrix}$$

By the corollary 6.32, the dual variables are obtained as :

$$\begin{pmatrix} y^*_1 \\ y^*_2 \end{pmatrix} = B^{-T} c_B = \begin{pmatrix} 0 & -1 \\ -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}.$$