CIVIL-557

Decision Aid Methodologies In Transportation

Lecture 3: Linear Programming and Duality

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On the previous lecture…

- Container Storage Inside a Container Terminal
  - Linear Programming
  - Simplex algorithm

Simplex method

How to find an initial basic feasible solution (BFS)?

$$
\begin{aligned}
&\text{Minimize} \quad && u_1^+ + u_1^- + u_2^+ + u_2^- \\
&\text{subject to} && \\
& && x_1 - u_1^+ + u_1^- = 4 \\
& && -x_2 + u_2^+ - u_2^- = 2 \\
& && x_1 + x_2 = 2 \\
& && x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{aligned}
$$

- Is there a feasible solution at all? (the problem might be infeasible)
- If so, how to find it?

Not always obvious…

Simplex method

Find an initial basic feasible solution (BFS) – the auxiliary problem

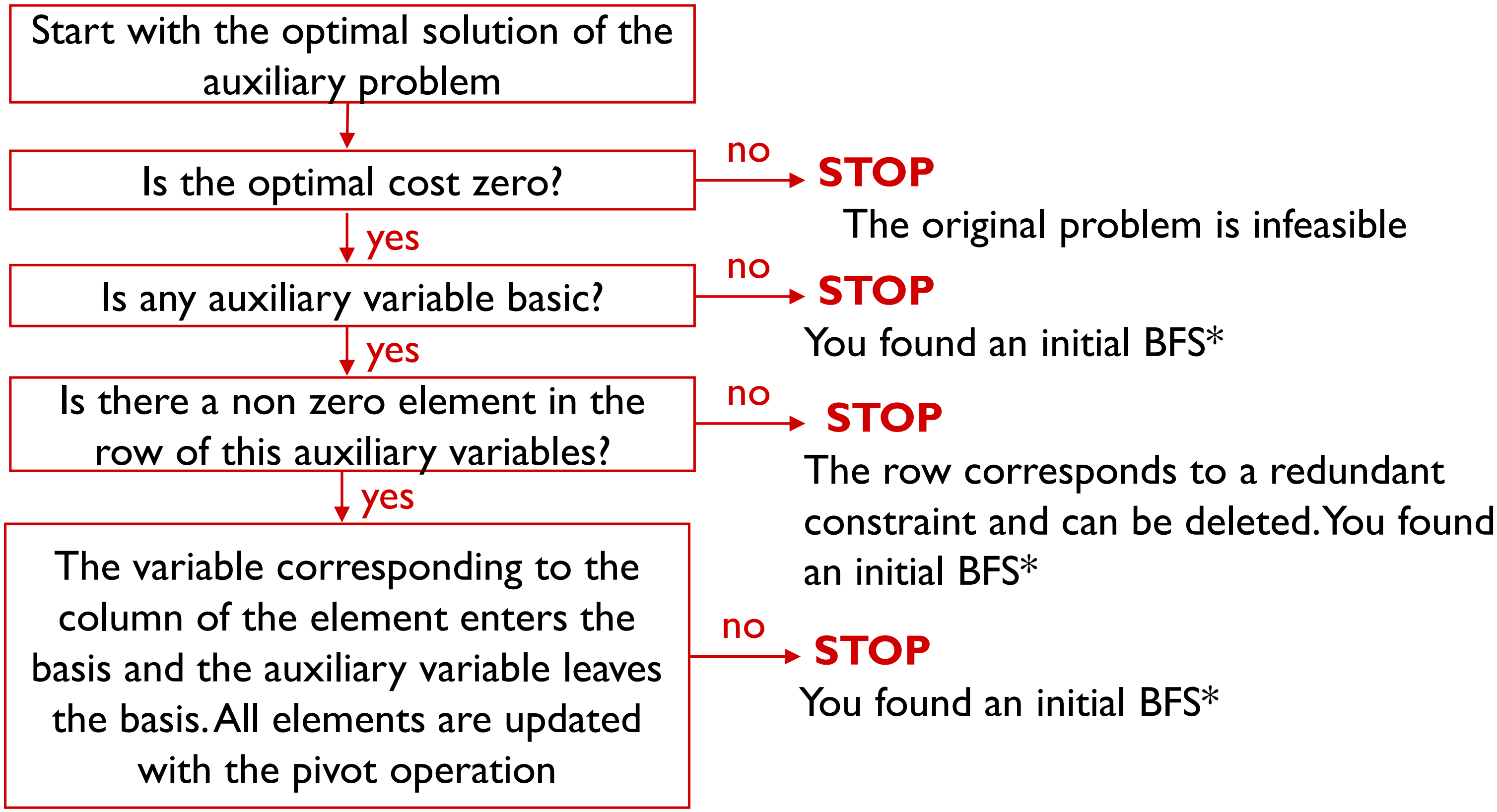
$$\begin{array}{ll} \textit{Minimize} & u_1^+ + u_1^- + u_2^+ + u_2^- \\ \textit{subject to} & x_1 - u_1^+ + u_1^- = 4 \\ & -x_2 + u_2^+ - u_2^- = 2 \\ & x_1 + x_2 = 2 \\ & x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0 \end{array}$$

$$\begin{array}{ll} \textit{Minimize} & a_1 + a_2 + a_3 \\ \textit{subject to} & x_1 - u_1^+ + u_1^- + a_1 = 4 \\ & -x_2 + u_2^+ - u_2^- + a_2 = 2 \\ & x_1 + x_2 + a_3 = 2 \\ & x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^-, a_1, a_2, a_3 \geq 0 \end{array}$$

- Introduce one auxiliary variable by constraint
- Replace the cost function by the sum of these auxiliary variables
- The original problem has a feasible solution **if and only if** the optimum value of the auxiliary problem is zero.
- The optimal solution of the auxiliary problem is used to construct the initial basic feasible solution of the original problem

Simplex method

Find an initial basic feasible solution (BFS) – the auxiliary problem



*compute the associated reduced costs and solve the initial problem with the simplex algorithm

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Simplex method

Back to our case study

Suppose 2 new containers are expected to arrive for storage in the next planning period of a terminal. Suppose there are only 2 blocks in the terminal, each with 20 storage spaces. For the moment, there are 6 containers in block 1 and 12 containers in block 2.

$$N = 2$$
$$B = 2$$
$$A = 20$$
$$a_1 = 6$$
$$a_2 = 12$$
$$F = \frac{6+12+2}{2\times 20} = 0.5$$

$$\begin{aligned} \textit{Minimize} \quad & u_1^+ + u_1^- + u_2^+ + u_2^- \\ \textit{subject to} \quad & \\ & 6 + x_1 - (u_1^+ - u_1^-) = 20 \times 0.5 \\ & 12 + x_2 - (u_2^+ - u_2^-) = 20 \times 0.5 \\ & x_1 + x_2 = 2 \\ & x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0 \end{aligned}$$

$$\begin{aligned} \textit{Minimize} \quad & u_1^+ + u_1^- + u_2^+ + u_2^- \\ \textit{subject to} \quad & \\ & x_1 - u_1^+ + u_1^- = 4 \\ & -x_2 + u_2^+ - u_2^- = 2 \\ & x_1 + x_2 = 2 \\ & x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0 \end{aligned}$$

$m = 3$ *basic variables*

Simplex method

Find an initial solution

$$\begin{array}{ll} \textit{Minimize} & a_1 + a_2 + a_3 \\ \textit{subject to} & \end{array}$$

$$x_1 - u_1^+ + u_1^- + a_1 = 4$$
$$-x_2 + u_2^+ - u_2^- + a_2 = 2$$
$$x_1 + x_2 + a_3 = 2$$
$$x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0$$

$n = 9$ variables

$m = 3$ constraints

3 basic variables

		x_1	x_2	u_1^+	u_1^-	u_2^+	u_2^-	a_1	a_2	a_3	
1	a_1	1	0	-1	1	0	0	1	0	0	4 /1
1	a_2	0	-1	0	0	1	-1	0	1	0	2 /0
1	a_3	1	1	0	0	0	0	0	0	1	2 /1
$\bar{c_j} = c_j - c_B^T B^{-1} A_j$		-2	0	1	-1	-1	1	0	0	0	8

a_3 leaves

x_1 enters

Are all reduced costs ≥ 0?

$$x_1 : \bar{c_j} = 0 - [1 \quad 1 \quad 1] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -2$$

Simplex method

Find an initial solution

Pivot column(p)		x_1	x_2	u_1^+	u_1^-	u_2^+	u_2^-	a_1	a_2	a_3		
	a_1	1	0	-1	1	0	0	1	0	0	4	
	a_2	0	-1	0	0	1	-1	0	1	0	2	Pivot row (q)
Pivot $T(q,p)$	a_3	1	1	0	0	0	0	0	0	1	2	
$\bar{c_j} = c_j - c_B^T B^{-1} A_j$		-2	0	1	-1	-1	1	0	0	0	8	a_3 leaves

x_1 enters

$$Pivot\ row:\ T(q,k) := \frac{T(q,k)}{T(q,p)} \qquad Other\ rows:\ T(i,k) := T(i,k) - \frac{T(i,p)T(q,k)}{T(q,p)}$$

		x_1	x_2	u_1^+	u_1^-	u_2^+	u_2^-	a_1	a_2	a_3		
1	a_1	0	-1	-1	1	0	0	1	0	-1	2	
1	a_2	0	-1	0	0	1	-1	0	1	0	2	a_1 leaves
0	x_1	1	1	0	0	0	0	0	0	1	2	
$\bar{c_j} = c_j - c_B^T B^{-1} A_j$		0	2	1	-1	-1	1	0	0	2	4	

u_1^- enters

Simplex method

Find an initial solution Pivot column(p) Pivot $T(q,p)$ Pivot row (q)

	x_1	x_2	u_1^+	u_1^-	u_2^+	u_2^-	a_1	a_2	a_3	
a_1	0	-1	-1	1	0	0	1	0	-1	2
a_2	0	-1	0	0	1	-1	0	1	0	2
x_1	1	1	0	0	0	0	0	0	1	2
$\bar{c_j} = c_j - c_B^T B^{-1} A_j$	0	2	1	-1	-1	1	0	0	2	4

a_1 leaves

u_1^- enters

$$Pivot\ row:\ T(q,k) \coloneqq \frac{T(q,k)}{T(q,p)} \qquad Other\ rows:\ T(i,k) \coloneqq T(i,k) - \frac{T(i,p)T(q,k)}{T(q,p)}$$

		x_1	x_2	u_1^+	u_1^-	u_2^+	u_2^-	a_1	a_2	a_3	
0	u_1^-	0	-1	-1	1	0	0	1	0	-1	2
1	a_2	0	-1	0	0	1	-1	0	1	0	2
0	x_1	1	1	0	0	0	0	0	0	1	2
	$\bar{c_j} = c_j - c_B^T B^{-1} A_j$	0	1	0	0	-1	1	1	0	1	2

a_2 leaves

u_2^+ enters

Simplex method

Find an initial solution

Pivot column(p) Pivot $T(q,p)$

	x_1	x_2	u_1^+	u_1^-	u_2^+	u_2^-	a_1	a_2	a_3	
u_1^-	0	-1	-1	1	0	0	1	0	-1	2
a_2	0	-1	0	0	1	-1	0	1	0	2
x_1	1	1	0	0	0	0	0	0	1	2
$\bar{c_j} = c_j - c_B^T B^{-1} A_j$	0	1	0	0	-1	1	1	0	1	2

Pivot row (q)

a_2 leaves

u_2^+ enters

$$Pivot\ row:\ T(q,k) := \frac{T(q,k)}{T(q,p)} \qquad Other\ rows:\ T(i,k) := T(i,k) - \frac{T(i,p)T(q,k)}{T(q,p)}$$

		x_1	x_2	u_1^+	u_1^-	u_2^+	u_2^-	a_1	a_2	a_3	
0	u_1^-	0	-1	-1	1	0	0	1	0	-1	2
0	u_2^+	0	-1	0	0	1	-1	0	1	0	2
0	x_1	1	1	0	0	0	0	0	0	1	2
	$\bar{c_j} = c_j - c_B^T B^{-1} A_j$	0	1	0	0	0	0	1	1	1	0

Optimal solution

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Simplex method

Find optimal solution

	x_1	x_2	u_1^+	u_1^-	u_2^+	u_2^-	a_1	a_2	a_3	
u_1^-	0	-1	-1	1	0	0	1	0	-1	2
u_2^+	0	-1	0	0	1	-1	0	1	0	2
x_1	1	1	0	0	0	0	0	0	1	2
$\bar{c_j} = c_j - c_B^T B^{-1} A_j$	0	1	0	0	0	0	1	1	1	0

Auxiliary variables are non basic

$$\begin{aligned} &Minimize\ u_1^+ + u_1^- + u_2^+ + u_2^- \\ &\text{s.t.}\quad x_1 - u_1^+ + u_1^- = 4 \\ &\quad -x_2 + u_2^+ - u_2^- = 2 \\ &\quad x_1 + x_2 = 2 \\ &\quad x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0 \end{aligned}$$

	x_1	x_2	u_1^+	u_1^-	u_2^+	u_2^-	
1 u_1^-	0	-1	-1	1	0	0	2
1 u_2^+	0	-1	0	0	1	-1	2
0 x_1	1	1	0	0	0	0	2
	0	2	2	0	0	2	4

$$\mathbf{x_1 = 2, x_2 = 0, u_1^+ = 0, u_1^- = 2,\ u_2^+ = 2, u_2^- = 0}$$

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Linear programming

$N = 2$
$B = 2$
$A = 20$
$a_1 = 6$
$a_2 = 12$
$F = \frac{6+12+2}{2\times 20} = 0.5$

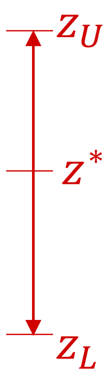
$$
\begin{array}{ll}
\textit{Minimize} & u_1^+ + u_1^- + u_2^+ + u_2^- \\
\textit{subject to} & \\
& x_1 - u_1^+ + u_1^- = 4 \\
& -x_2 + u_2^+ - u_2^- = 2 \\
& x_1 + x_2 = 2 \\
& x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{array}
$$

- $(x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^-) = (2, 0, 0, 2, 2, 0)$
- Can we test if $(x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^-) = (2, 0, 0, 2, 2, 0)$ is a feasible solution of the problem?
- Can we prove that $(x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^-) = (2, 0, 0, 2, 2, 0)$ is an optimal solution without solving the problem?
- Can we find lower and upper bounds on the optimal solution?

Bounds

- Upper bound
  - The objective value for any feasible solution is a upper bound on z^*

- Lower bound



$$
\begin{array}{ll}
\textit{Minimize} & z_L \leq \; u_1^+ + u_1^- + u_2^+ + u_2^- \; \leq z_U \\
\textit{subject to} & \\
 & x_1 - u_1^+ + u_1^- = 4 \\
 & -x_2 + u_2^+ - u_2^- = 2 \\
 & x_1 + x_2 = 2 \\
 & x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{array}
$$

How can we find a lower bound ?

Constraint relaxation

- General idea: incorporate constraints in the objective function

$$
\begin{aligned}
&\textit{Minimize} && u_1^+ + u_1^- + u_2^+ + u_2^- \\
&\textit{subject to} && \\
& && x_1 - u_1^+ + u_1^- = 4 \\
& && -x_2 + u_2^+ - u_2^- = 2 \\
& && x_1 + x_2 = 2 \\
& && x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{aligned}
$$

$z^* = 4$

$$
\begin{aligned}
&\textit{Minimize} && u_1^+ + u_1^- + u_2^+ + u_2^- \\
& && +\lambda_1(x_1 - u_1^+ + u_1^- - 4) \\
& && +\lambda_2(-x_2 + u_2^+ - u_2^- - 2) \\
& && +\lambda_3(x_1 + x_2 - 2)
\end{aligned}
$$

$$
\begin{aligned}
&\textit{subject to} \\
& x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{aligned}
$$

Constraint relaxation

- General idea: incorporate constraints in the objective function

$$
\begin{aligned}
&\textit{Minimize} \quad && u_1^+ + u_1^- + u_2^+ + u_2^- \\
&\textit{subject to} \\
& && x_1 - u_1^+ + u_1^- = 4 \\
& && -x_2 + u_2^+ - u_2^- = 2 \\
& && x_1 + x_2 = 2 \\
& && x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{aligned}
$$

$z^* = 4$

$$
\begin{aligned}
&\textit{Minimize} \quad && (1 - \lambda_1)u_1^+ + (1 + \lambda_1)u_1^- \\
& && +(1 + \lambda_2)u_2^+ + (1 - \lambda_2)u_2^- \\
& && +(\lambda_1 + \lambda_3)x_1 + (\lambda_3 - \lambda_2)x_2 \\
& && -4\lambda_1 - 2\lambda_2 - 2\lambda_3 \\
&\textit{subject to} \\
& && x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{aligned}
$$

(1) $\lambda_1 = \lambda_2 = \lambda_3 = 0$

(2) $\lambda_1 = 2, \lambda_2 = \lambda_3 = 0$

Lagrangian function

$$\begin{array}{ll} \textit{Minimize} & f(x) \\ \textit{subject to} & \\ & h(x) = 0 \\ & g(x) \leq 0 \end{array}$$

- Consider the vectors $\lambda \in R^m$, and $\mu \in R^p$
- The function

$$L(x, \lambda, \mu) = f(x) + \lambda^T h(x) + \mu^T g(x)$$
$$= f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{p} \mu_j g_j(x)$$

is called *Lagrangian* or the *Lagrangian* function

Dual function

$$L(x, \lambda, \mu) = f(x) + \lambda^T h(x) + \mu^T g(x)$$

$$= f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{p} \mu_j g_j(x)$$

- We can minimize the Lagrangian function for each fixed value of the parameters λ and μ
- The function that associates a set of parameters to the optimal value of the associated problem is called a *dual* function

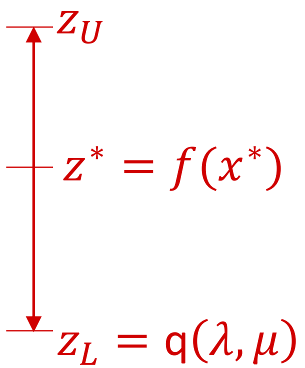
$$\mathrm{q}(\lambda, \mu) = \min_{x \in R^n} L(x, \lambda, \mu)$$

- The parameters λ, μ are called the dual variables, and the variables x are called primal variables

Bound from dual function

- Let x^* be the optimal solution of the initial problem, and let $q(\lambda, \mu)$ be the dual function to the same problem. Consider $\lambda \in R^m$, and $\mu \in R^p, \mu \geq 0$, then

$$q(\lambda, \mu) \leq f(x^*)$$



$$q(\lambda, \mu) = \min_{x \in R^n} L(x, \lambda, \mu)$$

$$q(\lambda, \mu) \leq L(x^*, \lambda, \mu)$$

$$q(\lambda, \mu) \leq f(x^*) + \sum_{i=1}^{m} \lambda_i h_i(x^*) + \sum_{j=1}^{p} \mu_j g_j(x^*)$$

$h(x^*) = 0$

$$q(\lambda, \mu) \leq f(x^*) + \sum_{j=1}^{p} \mu_j g_j(x^*)$$

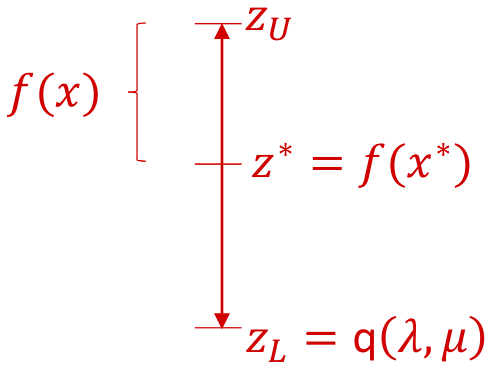
$g(x^*) \leq 0, \mu \geq 0$

$$q(\lambda, \mu) \leq f(x^*)$$

Bound from dual function

- Let x be the a feasible solution of the initial problem, and let $\mathrm{q}(\lambda, \mu)$ be the dual function to the same problem. Consider $\lambda \in R^m$, and $\mu \in R^p, \mu \geq 0$, then

$$\mathrm{q}(\lambda, \mu) \leq f(x)$$



Duality theory

Duality theory

- In the dual problem, the goal is to optimize the dual function, ensuring that the considered parameters λ and $\mu \geq 0$, do not generate an unbounded problem

(P) $\quad$ *Minimize* $\quad f(x)$

subject to

$$h(x) = 0$$
$$g(x) \leq 0$$

(D) $\quad$ *Maximize* $\quad q(\lambda, \mu)$

subject to

$$\mu \geq 0$$
$$(\lambda, \mu) \in X_q$$

with $X_q = \{\lambda, \mu \mid q(\lambda, \mu) > -\infty\}$

An example

(P) $\quad Minimize \quad 2x_1 + x_2$

$$x_1 + x_2 = 1$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

$Minimize \quad 2x_1 + x_2$

$$1 - x_1 - x_2 = 0 \quad h_1(x) \longrightarrow \lambda$$
$$-x_1 \leq 0 \quad g_1(x) \longrightarrow \mu_1$$
$$-x_2 \leq 0 \quad g_2(x) \longrightarrow \mu_2$$

$$L(x_1, x_2, \lambda, \mu_1, \mu_2) = 2x_1 + x_2 + \lambda(1 - x_1 - x_2) - \mu_1 x_1 - \mu_2 x_2$$
$$= (2 - \lambda - \mu_1)x_1 + (1 - \lambda - \mu_2)x_2 + \lambda$$

An example

$$L(x_1, x_2, \lambda, \mu_1, \mu_2) = (2 - \lambda - \mu_1)x_1 + (1 - \lambda - \mu_2)x_2 + \lambda$$

- In order for the dual function to be bounded, the coefficients of x_1 and x_2 have to be zero

$$2 - \lambda - \mu_1 = 0 \longrightarrow 2 - \lambda = \mu_1$$

$$1 - \lambda - \mu_2 = 0 \longrightarrow 1 - \lambda = \mu_2$$

- Since $\mu_1 \geq 0$, we need $\lambda \leq 2$
- Since $\mu_2 \geq 0$, we need $\lambda \leq 1$

$$X_q = \{\lambda, \mu_1, \mu_2 \,|\lambda \leq 1, \mu_1 \geq 0, \mu_2 \geq 0\}$$

- The dual function becomes

$$\mathrm{q}(\lambda, \mu_1, \mu_2) = \lambda$$

An example

(P) $\quad Minimize \quad 2x_1 + x_2$

$$x_1 + x_2 = 1$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

(D) $\quad Maximize \quad \lambda$

$$\lambda \leq 1$$
$$\mu_1 \geq 0$$
$$\mu_2 \geq 0$$

The dual problem

Primal (P)

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to $Ax \geq b$

$x \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

n variables

m constraints

Dual (D)

$$\max_{y \in \mathbb{R}^m} b^T y$$

subject to $A^T y \leq c$

$y \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

m variables

n constraints

Primal constraint	Dual variable	Primal variable	Dual constraint
$=$	$free$	≥ 0	$\leq$
$\leq$	≤ 0	≤ 0	$\geq$
$\geq$	≥ 0	$free$	$=$

Duality theory

Primal (P)

$$\min_{x \in \mathbb{R}^n} \; c^T x$$

subject to $\quad Ax \geq b$

$\quad x \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, b $\in \mathbb{R}^m$, c $\in \mathbb{R}^n$

Dual (D)

$$\max_{y \in \mathbb{R}^m} \; b^T y$$

subject to $\quad A^T y \leq c$

$\quad y \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, b $\in \mathbb{R}^m$, c $\in \mathbb{R}^n$

Weak duality

Let x^* be the optimal solution to the primal problem and let y^* be the optimal solution to the associated dual problem. Then $c^T x^* \geq \; b^T y^*$

- The dual function provides lower bounds on the optimal value of the problem

Duality theory

Primal (P)

$$\min_{x \in \mathbb{R}^n} \; c^T x$$

subject to $Ax \geq b$

$x \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, b $\in \mathbb{R}^m$, c $\in \mathbb{R}^n$

Dual (D)

$$\max_{y \in \mathbb{R}^m} \; b^T y$$

subject to $A^T y \leq c$

$y \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, b $\in \mathbb{R}^m$, c $\in \mathbb{R}^n$

Strong duality

Consider a linear optimization problem and its dual. If one problem has an optimal solution, so does the other one, and the optimal value of their objective functions are the same.

- It can be shown that **strong duality always holds for LPs**

Duality theory

Primal (P)

$$\min_{x \in \mathbb{R}^n} \; c^T x$$

subject to

$$Ax \geq b$$
$$x \geq 0,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

Dual (D)

$$\max_{y \in \mathbb{R}^m} \; b^T y$$

subject to

$$A^T y \leq c$$
$$y \geq 0,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

		Dual		
		Optimal Solution	Unbounded	Infeasible
Primal	Optimal Solution			
	Unbounded			
	Infeasible			

Complementary slackness

Primal (P)

$$\min_{x \in \mathbb{R}^n} \; c^T x$$

subject to $Ax \geq b$

$x \geq 0,$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

Dual (D)

$$\max_{y \in \mathbb{R}^m} \; b^T y$$

subject to $A^T y \leq c$

$y \geq 0,$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

- Strong duality implies that assuming a solution x^* for (P) and a solution y^* for (D):

  - If $x_j^* > 0$, then the jth constraint in (D) is binding
  - If the jth constraint in (D) is not binding, then $x_j^* = 0$
  - If $y_i^* > 0$, then the ith constraint in (P) is binding
  - If the ith constraint in (P) is not binding, then $y_i^* = 0$

Complementary slackness conditions

Duality theory

Minimize $u_1^+ + u_1^- + u_2^+ + u_2^-$

Subject to
$$\begin{aligned} x_1 - u_1^+ + u_1^- &= 4 \\ -x_2 + u_2^+ - u_2^- &= 2 \\ x_1 + x_2 &= 2 \\ x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- &\geq 0 \end{aligned}$$

Maximize $4\,y_1 + 2\,y_2 + 2\,y_3$

Subject to
$$\begin{aligned} y_1 + y_3 &\leq 0 \\ -y_2 + y_3 &\leq 0 \\ -y_1 &\leq 1 \\ y_1 &\leq 1 \\ y_2 &\leq 1 \\ -y_2 &\leq 1 \end{aligned}$$

- Is $(x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^-) = (2, 0, 0, 2, 2, 0)$ an optimal solution?
- The constraints $(1), (4),$ and (5) in (D) are binding:

  $(4)\ y_1 = 1 \qquad (5)\ y_2 = 1 \qquad (1)\ y_1 + y_3 = 0 \longleftrightarrow y_3 = -1$

- Is $(y_1, y_2, y_3) = (1,1,-1)$ a feasible solution for (P)?
- Are both objective function values equal ?

Economic interpretation of duality

- Is there any special meaning of the dual variables?
- What is a dual problem trying to do?
- What's the role of the complementary slackness conditions in decision making?

Economic interpretation of duality

- Any optimization problem can be seen from two points of view:

    1. From the viewpoint of the one solving the problem
    2. From the viewpoint of the one who defines the rules of the game

There are those who are subject to constraints and others who impose them

Economic interpretation of duality

- The transportation problem:
  - A manufacturer makes a single product in factories at m locations, and wishes to ship them to n distribution centers.
  - Each factory i makes s_i units of this product, and each distribution center j has a demand d_j for the product, with $\sum_i s_i \geq \sum_j d_j$.
  - Further, there is a cost c_{ij} for shipping each unit of the product from factory i to distribution center j.
  - The manufacturer wishes to determine a shipping schedule that ships from available supply to satisfy demand and has minimum total shipping cost

Economic interpretation of duality

- Mathematical model:
  - Let x_{ij} represent the amount shipped from factory i to distribution center j

Minimize
$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to
$$\sum_{j=1}^{n} x_{ij} \le s_i, \qquad i = 1, \ldots m$$
$$\sum_{i=1}^{m} x_{ij} \ge d_j, \qquad j = 1, \ldots n$$
$$x_{ij} \ge 0, \qquad i = 1, \ldots m, j = 1, \ldots n$$

Economic interpretation of duality

- Mathematical model:
  - Let write the LP in canonical min form

Minimize
$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to
$$-\sum_{j=1}^{n} x_{ij} \geq -s_i, \qquad i = 1, \dots m \qquad p_i$$
$$\sum_{i=1}^{m} x_{ij} \geq d_j, \qquad j = 1, \dots n \qquad q_i$$
$$x_{ij} \geq 0, \qquad i = 1, \dots m, j = 1, \dots n$$

Economic interpretation of duality

- Mathematical model:
  - The dual is

Maximize
$$\sum_{j=1}^{n} d_j q_j - \sum_{i=1}^{m} s_i p_i$$

Subject to
$$q_j - p_i \leq c_{ij}, \quad i = 1, \dots m, j = 1, \dots n$$

$$p_i, q_j \geq 0, \quad i = 1, \dots m, j = 1, \dots n$$

Economic interpretation of duality

- Interpretation:
  - The marketplace will naturally make transportation arrangements by decoupling the manufacturer from the distribution centers, and having an independent trucking company serve as the middleman between these centers
  - The trucker will buy all of the product made at each manufacturing center i for price p_i, and then will sell all of the product demanded by distribution center j for price q_j

Economic interpretation of duality

- Interpretation:

  - The manufacturers and distribution centers are willing to accept this arrangement so long as the net loss of value along each route (buy-back price – selling price) does not exceed the cost of shipping along that particular route

  - The Trucker's objective in setting prices is to maximize profits while staying competitive with current transportation costs

Economic interpretation of duality

- Conclusion:

  - The **profit** the trucker gets from taking over the transportation portion of the process **is exactly the same** as the **cost** the manufacturer would incur by doing it in-house.

Main references

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