EPFL
ENAC TRANSP-OR
Prof. M. Bierlaire

Mathematical Modeling of Behavior
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EXERCISE SESSION 6 (Solution)

Exercise 1 Consider a logit model for mode choice with three alternatives: car (c), public transportation (p) and slow modes (s). The utility specifications are the following:

$$\begin{aligned} U_{c,n} &= \beta_{c,1} + \beta_{c,2} \cdot \text{cost}_{c,n} + \beta_{c,3} \cdot \text{tt}_{c,n} + \varepsilon_{c,n}, \\ U_{p,n} &= \beta_{p,1} + \beta_{p,2} \cdot \text{cost}_{p,n} + \beta_{p,3} \cdot \text{tt}_{p,n} + \varepsilon_{p,n}, \\ U_{s,n} &= \beta_{s,1} + \beta_{s,3} \cdot \text{tt}_{s,n} + \varepsilon_{s,n}, \end{aligned}$$

where $\text{cost}_{i,n}$ is the cost associated by customer n with alternative $i \in \{c, p\}$, and $\text{tt}_{i,n}$ is the travel time associated with alternative $i \in \{c, p, s\}$ by customer n. We denote by $\mathbb{E}[\varepsilon_{i,n}] = \alpha_i$ the mean of the distribution of the error terms $\varepsilon_{i,n}$ $\forall i \in \{c, p, s\}, n$.

Show that it is possible to rewrite the utility functions in order to have $\mathbb{E}[\varepsilon_{i,n}] = 0\ \forall i \in \{c, p, s\}, n$ and the same probabilities.

Solution: First we need to perform a change of variable by adding and subtracting α_i to each utility function:

$$\begin{aligned} U_{c,n} &= (\beta_{c,1} + \alpha_c) + \beta_{c,2} \cdot \text{cost}_{c,n} + \beta_{c,3} \cdot \text{tt}_{c,n} + (\varepsilon_{c,n} - \alpha_c), \\ U_{p,n} &= (\beta_{p,1} + \alpha_p) + \beta_{p,2} \cdot \text{cost}_{p,n} + \beta_{c,3} \cdot \text{tt}_{p,n} + (\varepsilon_{p,n} - \alpha_p), \\ U_{s,n} &= (\beta_{s,1} + \alpha_s) + \beta_{c,3} \cdot \text{tt}_{s,n} + (\varepsilon_{s,n} - \alpha_s), \end{aligned}$$

In this way, $\mathbb{E}[\varepsilon_{i,n} - \alpha_i] = 0$.

With respect to the choice probabilities, we define a new notation for the changes of variable: $U'_{i,n} = V'_{i,n} + \varepsilon'_{i,n}$, where $V'_{i,n} = V_{i,n} + \alpha_i$ and $\varepsilon'_{i,n} = \varepsilon_{i,n} - \alpha_i$, $\forall i \in \{c, p, s\}, n$. Then, the following proves that both models provide the same probabilities:

$$
\begin{aligned}
\mathrm{P}(U^{'}_{i,n} \geq U^{'}_{j,n}) &= \mathrm{P}(V^{'}_{i,n} + \varepsilon^{'}_{i,n} \geq V^{'}_{j,n} + \varepsilon^{'}_{j,n}) \\
&= \mathrm{P}(V^{'}_{i,n} + \varepsilon^{'}_{i,n} \geq V^{'}_{j,n} + \varepsilon^{'}_{j,n}) \\
&= \mathrm{P}(V_{i,n} + \alpha_i + \varepsilon_{i,n} - \alpha_i \geq V_{j,n} + \alpha_j + \varepsilon_{j,n} - \alpha_j) \\
&= \mathrm{P}(V_{i,n} + \varepsilon_{i,n} \geq V_{j,n} + \varepsilon_{j,n}) \\
&= \mathrm{P}(U_{i,n} \geq U_{j,n})
\end{aligned}
$$

Exercise 2 Consider the logit model from the previous exercise and define the following additional specifications (use the model from exercise 1 as a base model for each new specification). For each specification, indicate if it is linear-in-parameters and if it is linear in the involved variables.

1. Propose a linear-in-parameters specification and a non linear-in-parameters specification that captures that the marginal effect of travel time in the utility varies with time.

   Solution: One example of a linear-in-parameter specification is

   $$U_{i,n} = \cdots + \beta_{i,3} \cdot \ln(\mathrm{tt}_{i,n}) + \ldots \;\forall i \in \{c, p, s\}, n.$$

   A not linear-in-parameter specification can be defined with a Box-Cox transformation of the variable $\mathrm{tt}_{i,n}$:

   $$\mathrm{tt}^{(\lambda)}_{i,n} = \begin{cases} \frac{\mathrm{tt}^{\lambda}_{i,n} - 1}{\lambda} & \text{if } \lambda \neq 0, \\ \ln(\mathrm{tt}_{i,n}) & \text{if } \lambda = 0, \end{cases}$$

   where λ is an unknown parameter to be estimated. The resulting utility function is

   $$U_{i,n} = \cdots + \beta_{i,3} \cdot \mathrm{tt}^{(\lambda)}_{i,n} + \ldots \;\forall i \in \{c, p, s\}, n.$$

   Both specifications are nonlinear in $\mathrm{tt}_{i,n}$.

2. Propose a specification that captures a variation in the sensitivity towards travel time for trips made by car when its length is classified as short ($\mathrm{tt}_{c,n} \leq 20$ minutes), medium ($20 < \mathrm{tt}_{c,n} \leq 60$ minutes) and long ($\mathrm{tt}_{c,n} > 60$ minutes).

   Solution: We consider a piecewise linear specification. For each of the above-mentioned intervals, we define the following variables:

$$\mathrm{tt}_{c,n}^{\mathrm{short}} = \begin{cases} \mathrm{tt}_{c,n} & \text{if } \mathrm{tt}_{c,n} < 20 \\ 20 & \text{if } \mathrm{tt}_{c,n} \geq 20 \end{cases} \tag{1}$$

$$\mathrm{tt}_{c,n}^{\mathrm{medium}} = \begin{cases} 0 & \text{if } \mathrm{tt}_{c,n} < 20 \\ \mathrm{tt}_{c,n} - 20 & \text{if } 20 \leq \mathrm{tt}_{c,n} < 60 \\ 40 & \text{if } \mathrm{tt}_{c,n} \geq 60 \end{cases} \tag{2}$$

$$\mathrm{tt}_{c,n}^{\mathrm{long}} = \begin{cases} 0 & \text{if } \mathrm{tt}_{c,n} < 60 \\ \mathrm{tt}_{c,n} - 60 & \text{if } \mathrm{tt}_{c,n} \geq 60 \end{cases} \tag{3}$$

We define a parameter for each of these variables and we modify the utility function for car as follows:

$$U_{c,n} = \cdots + \beta_{c,3}^{\mathrm{short}} \cdot \mathrm{tt}_{c,n}^{\mathrm{short}} + \beta_{c,3}^{\mathrm{medium}} \cdot \mathrm{tt}_{c,n}^{\mathrm{medium}} + \beta_{c,3}^{\mathrm{long}} \cdot \mathrm{tt}_{c,n}^{\mathrm{long}} + \ldots.$$

This specification is linear-in-parameters and nonlinear in $\mathrm{tt}_{c,n}$.

3. Define a dummy variable for individuals owning a travel card in order to propose a specification that assumes that owning a travel card might have an impact on the choice.

   Solution: We define the dummy variable $\mathrm{travel\_card}_n$, which is 1 if individual n owns a travel card and 0 otherwise. One example of such specification is

   $$U_{p,n} = \cdots + \beta_{\mathrm{travel\_card}} \cdot \mathrm{travel\_card}_n + \ldots,$$

   where $\beta_{\mathrm{travel\_card}}$ is the parameter associated with $\mathrm{travel\_card}_n$. In this specification we assume that the utility of the public transportation alternative is the only one including this term. Notice that the same term could have been included in another utility function (e.g., car), but the interpretation of the parameter would be different. In general, these should be interacted with all ASCs. This case is specific, and motivated by behavioral assumptions. This specification is both linear-in-parameters and linear in $\mathrm{travel\_card}_n$.

4. Assume that the variable age_n represents the age of individual n. Propose a specification that captures the fact that the travel time of slow modes varies continuously with the age.

   Solution: We can define such an interaction linearly or nonlinearly. We define a parameter $\hat{\beta}_{s,3}$ that represents the sensitivity of the utility function to the variable $\mathrm{tt}_{s,n}$ when $\mathrm{age}_n = \mathrm{age}_n^{\mathrm{ref}}$, where $\mathrm{age}_n^{\mathrm{ref}}$ is an arbitrary reference value of the age (e.g., the mean, the mode, the maximum). The utility of slow modes is written as

   $$U_{s,n} = \cdots + \hat{\beta}_{s,3} \cdot \frac{\mathrm{age}_n}{\mathrm{age}_n^{\mathrm{ref}}} \cdot \mathrm{tt}_{s,n} + \ldots$$

in the linear case and as

$$U_{s,n} = \dots + \hat{\beta}_{s,3} \cdot \left(\frac{\text{age}_n}{\text{age}_n^{\text{ref}}}\right)^{\lambda} \cdot \text{tt}_{s,n} + \dots$$

in the nonlinear case, where λ represents the elasticity of $\left(\frac{\text{age}_n}{\text{age}_n^{\text{ref}}}\right)^{\lambda}$ with respect of variations of age_n. Both $\hat{\beta}_{s,3}$ and λ are unknown parameters to be estimated. The first specification is linear-in-parameters and the second one is not. Both are nonlinear in the variable $\text{tt}_{s,n}$.

Exercise 3 Answer to the following questions.

1. Describe the Independence from Irrelevant Alternatives (IIA) property. Under which circumstance is the IIA property violated? (There are different circumstances, please describe here the one discussed in the lecture.)

   Solution: The IIA property can be stated in different ways. One statement is that the relative choice probabilities between any two alternatives is independent of the other available alternatives. Another statement says that the choice probabilities from a subset of alternatives is dependent only on the alternatives included in this subset and is independent of any other alternatives that may exist. As seen in the lecture, this property is violated when the alternatives share unobserved attributes (e.g., the fact that the blue bus alternative and the red bus alternative are essentially the same), so the error terms are correlated (which violates the independence assumption).

2. Recall the red bus/ blue bus paradox that has been seen in the lecture. Travelers initially face a decision between two modes of transportation: car and blue bus. The travel times of both modes, car and blue bus, are equal. Travel time is also the only variable considered in the utility. Then, we suppose that a third mode, namely the red bus, is introduced and that the travelers consider it to be exactly the same as the blue bus. Assume that the error terms for the red and blue bus are correlated and that the correlation is 95%. Derive the scale parameter (μ_m) and calculate the probabilities of choosing car and bus[1].

   Solution: Given that the correlation is 95%, μ_m is computed as:

   $$1 - \mu^2/\mu_m^2 = 0.95 \Leftrightarrow \underline{\mu_m = \sqrt{1/0.05}} \tag{4}$$

   The probabilities of choosing car and bus can be obtained by using the NL model. The expected maximum utility of bus is:

   $$\begin{aligned} V_{\text{bus}} &= \frac{1}{\mu_m} \ln(e^{\mu_m \beta T} + e^{\mu_m \beta T}) \\ &= \beta T + \frac{1}{\mu_m} \ln 2 \end{aligned} \tag{5}$$

[1] Note that μ is normalized to one.

where T is the travel time, and β is its coefficient. The probability of choosing car is:

$$\begin{aligned} P(\text{car}) &= \frac{e^{\mu V_{\text{car}}}}{e^{\mu V_{\text{car}}} + e^{\mu V_{\text{bus}}}} \\ &= \frac{e^{\beta T}}{e^{\beta T} + e^{\beta T + \frac{1}{\mu_m}\ln 2}} \\ &= \frac{1}{1 + 2^{1/\mu_m}} \end{aligned} \tag{6}$$

Given the value of μ_m, finally we obtain:

$$\underline{P(\text{car})} = \frac{1}{1 + 2^{\sqrt{0.05}}} = \underline{0.461} \tag{7}$$

and

$$\underline{P(\text{bus})} = 1 - P(\text{car}) = \underline{0.539} \tag{8}$$

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