Theoretical foundations – 2.4 Random utility theory

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Solution of the practice quiz.

The CDF of the error terms is given by

$$F_\varepsilon(\varepsilon_i, \varepsilon_j) = e^{-e^{-\varepsilon_i}} e^{-e^{-\varepsilon_j}}. \tag{1}$$

We have

$$P(i|\{i,j\}) = \int_{\varepsilon=-\infty}^{+\infty} \frac{\partial F_\varepsilon}{\partial \varepsilon_i}(\varepsilon, V_i - V_j + \varepsilon) d\varepsilon. \tag{2}$$

From (1), we have

$$\frac{\partial F_\varepsilon}{\partial \varepsilon_i}(\varepsilon_i, \varepsilon_j) = e^{-e^{-\varepsilon_i}} e^{-e^{-\varepsilon_j}} e^{-\varepsilon_i}. \tag{3}$$

Therefore,

$$\frac{\partial F_\varepsilon}{\partial \varepsilon_i}(\varepsilon, V_i - V_j + \varepsilon) = e^{-e^{-\varepsilon}} e^{-e^{-(V_i - V_j + \varepsilon)}} e^{-\varepsilon} = e^{-e^{-\varepsilon}} e^{-Ke^{-\varepsilon}} e^{-\varepsilon} \tag{4}$$

where

$$K = \exp(-(V_i - V_j)). \tag{5}$$

Therefore,

$$P(i|\{i,j\}) = \int_{\varepsilon=-\infty}^{+\infty} e^{-e^{-\varepsilon}} e^{-Ke^{-\varepsilon}} e^{-\varepsilon} d\varepsilon. \tag{6}$$

Define

$$t = -e^{-\varepsilon},\ dt = e^{-\varepsilon} d\varepsilon,$$

to obtain

$$P(i|\{i,j\}) = \int_{t=-\infty}^{0} e^{(1+K)t} dt = \frac{1}{1+K}. \tag{7}$$

Using (5), we obtain the simple expression:

$$P(i|\{i,j\}) = \frac{1}{1+\exp(-(V_i - V_j))} = \frac{e^{V_i}}{e^{V_i} + e^{V_j}}. \tag{8}$$

This happens to be the binary logit model.