

# Mixture Models — Simulation-based Estimation

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# Outline

- 1 Mixtures
- 2 Relaxing the independence assumption
- 3 Relaxing the identical distribution assumption
- 4 Taste heterogeneity
- 5 Latent classes
- 6 Summary

# Mixtures

## Mixture probability distribution function

Convex combination of other probability distribution functions.

### Property

- Let  $f(\varepsilon, \theta)$  be a parametrized family of distribution functions
- Let  $w(\theta)$  be a non negative function such that

$$\int_{\theta} w(\theta) d\theta = 1$$

- Then

$$g(\varepsilon) = \int_{\theta} w(\theta) f(\varepsilon, \theta) d\theta$$

is also a distribution function.

# Mixtures

We say that  $g$  is a  $w$ -mixture of  $f$

- If  $f$  is a logit model,  $g$  is a **continuous  $w$ -mixture of logit**
- If  $f$  is a MEV model,  $g$  is a **continuous  $w$ -mixture of MEV**

# Mixtures

## Discrete mixtures

If  $w_i$ ,  $i = 1, \dots, n$  are non negative weights such that

$$\sum_{i=1}^n w_i = 1$$

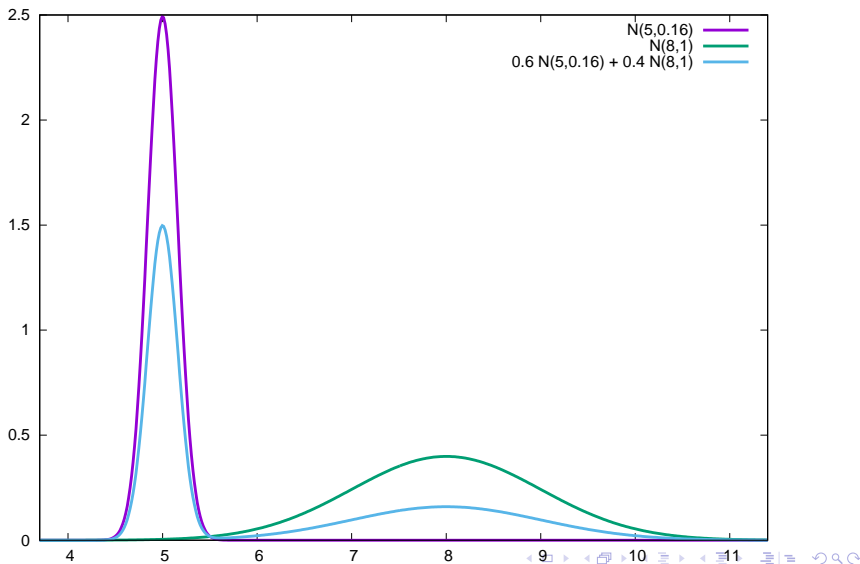
then

$$g(\varepsilon) = \sum_{i=1}^n w_i f(\varepsilon, \theta_i)$$

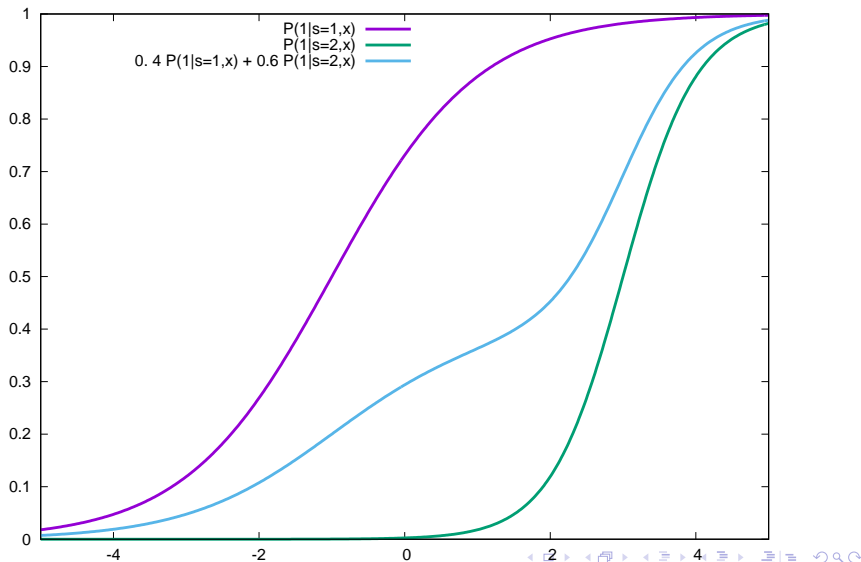
is also a distribution function where  $\theta_i$ ,  $i = 1, \dots, n$  are parameters.

We say that  $g$  is a discrete  $w$ -mixture of  $f$ .

# Example: discrete mixture of normal distributions



# Example: discrete mixture of binary logit models



# Mixtures

## General motivation

Generate flexible distributional forms

## For discrete choice

- correlation across alternatives
- alternative specific variances
- taste heterogeneity
- ...



# Continuous Mixtures of logit

Combining probit and logit

Error components

$$U_{in} = V_{in} + \xi_{in} + \nu_{in}$$

i.i.d EV (logit): tractability

Normal distribution (probit): flexibility

# Logit

## Specification of the utility functions

$$\begin{aligned}U_{\text{auto}} &= \beta X_{\text{auto}} + \nu_{\text{auto}} \\U_{\text{bus}} &= \beta X_{\text{bus}} + \nu_{\text{bus}} \\U_{\text{subway}} &= \beta X_{\text{subway}} + \nu_{\text{subway}}\end{aligned}$$

## Distributional assumption

$\nu$  i.i.d. extreme value

## Choice model

$$\Pr(\text{auto}|X, \mathcal{C}) = \frac{e^{\beta X_{\text{auto}}}}{e^{\beta X_{\text{auto}}} + e^{\beta X_{\text{bus}}} + e^{\beta X_{\text{subway}}}}$$

# Normal mixture of logit

## Specification of the utility functions

$$\begin{aligned}
 U_{\text{auto}} &= \beta X_{\text{auto}} + \xi_{\text{auto}} + \nu_{\text{auto}} \\
 U_{\text{bus}} &= \beta X_{\text{bus}} + \xi_{\text{bus}} + \nu_{\text{bus}} \\
 U_{\text{subway}} &= \beta X_{\text{subway}} + \xi_{\text{subway}} + \nu_{\text{subway}}
 \end{aligned}$$

## Distributional assumptions

- $\nu$  i.i.d. extreme value
- $\xi \sim N(0, \Sigma)$

## Choice model

$$\Pr(\text{auto}|X, \xi) = \frac{e^{\beta X_{\text{auto}} + \xi_{\text{auto}}}}{e^{\beta X_{\text{auto}} + \xi_{\text{auto}}} + e^{\beta X_{\text{bus}} + \xi_{\text{bus}}} + e^{\beta X_{\text{subway}} + \xi_{\text{subway}}}}$$

$$P(\text{auto}|X) = \int_{\xi} \Pr(\text{auto}|X, \xi) f(\xi) d\xi$$

# Calculation

## Choice model

$$P(\text{auto}|X) = \int_{\xi} \Pr(\text{auto}|X, \xi) f(\xi) d\xi$$

## Calculation

- Integral has no closed form.
- If one dimension is involved, numerical integration can be used.
- With more dimensions, Monte Carlo simulation must be used.

# Simulation

In order to approximate

$$P(\text{auto}|X) = \int_{\xi} \Pr(\text{auto}|X, \xi) f(\xi) d\xi$$

- Draw from  $f(\xi)$  to obtain  $r_1, \dots, r_R$
- Compute  $P(\text{auto}|X) \approx \tilde{P}(\text{auto}|X) = \frac{1}{R} \sum_{k=1}^R P(\text{auto}|X, r_k) =$

$$\frac{1}{R} \sum_{k=1}^R \frac{e^{\beta X_{\text{auto}} + r_{1k}}}{e^{\beta X_{\text{auto}} + r_{1k}} + e^{\beta X_{\text{bus}} + r_{2k}} + e^{\beta X_{\text{subway}} + r_{3k}}}$$

# Simulation

Can approximate as close as needed

$$P(\text{auto}|X) = \lim_{R \rightarrow \infty} \frac{1}{R} \sum_{k=1}^R P(\text{auto}|X, r_k).$$

In practice

- Efficient methods to draw from the distribution.
- $R$  must be large enough.

# Outline

- 1 Mixtures
- 2 Relaxing the independence assumption
  - Nesting
  - Cross-nesting
- 3 Relaxing the identical distribution assumption
- 4 Taste heterogeneity
- 5 Latent classes
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# Capturing correlations: nesting

## Specification of the utility functions

$$\begin{aligned}
 U_{\text{auto}} &= \beta X_{\text{auto}} && + \nu_{\text{auto}} \\
 U_{\text{bus}} &= \beta X_{\text{bus}} &+ \sigma_{\text{transit}} \eta_{\text{transit}} &+ \nu_{\text{bus}} \\
 U_{\text{subway}} &= \beta X_{\text{subway}} &+ \sigma_{\text{transit}} \eta_{\text{transit}} &+ \nu_{\text{subway}}
 \end{aligned}$$

## Distributional assumptions

- $\nu$  i.i.d. extreme value,
- $\eta_{\text{transit}} \sim N(0, 1)$ ,  $\sigma_{\text{transit}}^2 = \text{cov}(\text{bus}, \text{subway})$

## Choice model

$$\Pr(\text{auto} | X, \eta_{\text{transit}}) = \frac{e^{\beta X_{\text{auto}}}}{e^{\beta X_{\text{auto}}} + e^{\beta X_{\text{bus}} + \sigma_{\text{transit}} \eta_{\text{transit}}} + e^{\beta X_{\text{subway}} + \sigma_{\text{transit}} \eta_{\text{transit}}}}$$

$$P(\text{auto} | X) = \int \Pr(\text{auto} | X, \eta) f(\eta) d\eta$$



# Nesting structure

## Example: residential telephone

	Ct. BM	Ct. SM	Ct. LF	Ct. EF	$\beta_C$	$\sigma_M$	$\sigma_F$
BM	1	0	0	0	$\ln(\text{cost}(\text{BM}))$	$\eta_M$	0
SM	0	1	0	0	$\ln(\text{cost}(\text{SM}))$	$\eta_M$	0
LF	0	0	1	0	$\ln(\text{cost}(\text{LF}))$	0	$\eta_F$
EF	0	0	0	1	$\ln(\text{cost}(\text{EF}))$	0	$\eta_F$
MF	0	0	0	0	$\ln(\text{cost}(\text{MF}))$	0	$\eta_F$

# Nesting structure

## Identification issues

- If there are two nests, only one  $\sigma$  is identified
- If there are more than two nests, all  $\sigma$ 's are identified

Walker (2001)

## Results with 5000 draws

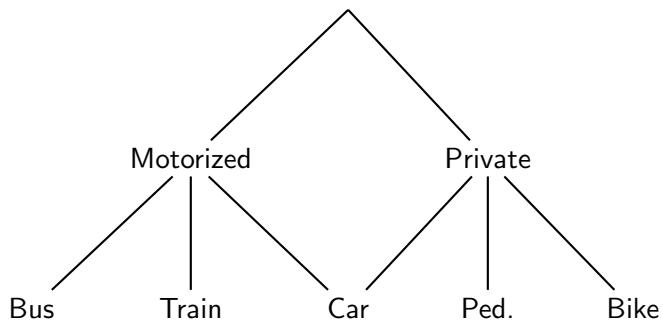
$\mathcal{L}$	NL		NML		NML $\sigma_F = 0$		NML $\sigma_M = 0$		NML $\sigma_F = \sigma_M$	
	Estim.	Scaled	Estim.	Scaled	Estim.	Scaled	Estim.	Scaled	Estim.	Scaled
	-473.219		-472.768		-473.146		-472.779		-472.846	
Ct. BM	-1.78	1.00	-3.81	1.00	-3.79	1.00	-3.81	1.00	-3.81	1.00
Ct. EF	-0.558	0.313	-1.20	0.314	-1.19	0.313	-1.20	0.314	-1.20	0.314
Ct. LF	-0.512	0.287	-1.10	0.287	-1.09	0.287	-1.09	0.287	-1.09	0.287
Ct. SM	-1.41	0.788	-3.02	0.791	-3.00	0.790	-3.01	0.791	-3.02	0.791
$\beta_C$	-1.49	0.835	-3.26	0.855	-3.24	0.855	-3.26	0.855	-3.26	0.854
$\mu_{\text{FLAT}}$	2.29									
$\mu_{\text{MEAS}}$	2.06									
$\sigma_F$			3.02		0.00		3.06		2.17	
$\sigma_M$			0.530		3.02		0.00		2.17	
$\sigma_F^2 + \sigma_M^2$			9.40		9.15		9.37		9.43	

# Comments

- The scale of the parameters is different between NL and the mixture model
- Normalization can be performed in several ways
  - $\sigma_F = 0$
  - $\sigma_M = 0$
  - $\sigma_F = \sigma_M$
- Final log likelihood should be the same
- But... estimation relies on simulation
- Only an approximation of the log likelihood is available
- Final log likelihood with 50000 draws:

Unnormalized:	-472.872	$\sigma_M = \sigma_F$ :	-472.875
$\sigma_F = 0$ :	-472.884	$\sigma_M = 0$ :	-472.901

# Cross nesting



# Cross nesting

## Specification

$$\begin{aligned}
 U_{\text{bus}} &= V_{\text{bus}} + \xi_1 && + \nu_{\text{bus}} \\
 U_{\text{train}} &= V_{\text{train}} + \xi_1 && + \nu_{\text{train}} \\
 U_{\text{car}} &= V_{\text{car}} + \xi_1 + \xi_2 && + \nu_{\text{car}} \\
 U_{\text{ped}} &= V_{\text{ped}} && + \xi_2 + \nu_{\text{ped}} \\
 U_{\text{bike}} &= V_{\text{bike}} && + \xi_2 + \nu_{\text{bike}}
 \end{aligned}$$

## Choice model

$$P(\text{car}) = \int_{\xi_1} \int_{\xi_2} P(\text{car} | \xi_1, \xi_2) f(\xi_1) f(\xi_2) d\xi_2 d\xi_1$$

# Identification issue

- Not all parameters can be identified
- For logit, one ASC has to be constrained to zero
- Identification of NML is important and tricky
- See Walker, Ben-Akiva & Bolduc (2007) for a detailed analysis

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## Alternative specific variance

Logit: i.i.d. error terms

- In particular, they have the same variance

$$U_{in} = \beta^T x_{in} + ASC_i + \nu_{in}$$

- $\nu_{in}$  i.i.d.  $EV(0, \mu) \Rightarrow \text{Var}(\nu_{in}) = \pi^2/6\mu^2$

Relax the identical distribution assumption

$$U_{in} = \beta^T x_{in} + ASC_i + \sigma_i \xi_i + \nu_{in}$$

where  $\xi_i \sim N(0, 1)$

Variance

$$\text{Var}(\sigma_i \xi_i + \nu_{in}) = \sigma_i^2 + \frac{\pi^2}{6\mu^2}$$

# Alternative specific variance

## Identification issue

- Not all  $\sigma$ s are identified
- One of them must be constrained to zero
- Not necessarily the one associated with the ASC constrained to zero
- In theory, the smallest  $\sigma$  must be constrained to zero
- In practice, we don't know a priori which one it is
- Solution:
  - 1 Estimate a model with a full set of  $\sigma$ s
  - 2 Identify the smallest one and constrain it to zero.

# Alternative specific variance

## Example with Swissmetro

	ASC_CAR	ASC_SBB	ASC_SM	B_COST	B_FR	B_TIME
Car	1	0	0	cost	0	time
Train	0	0	0	cost	freq.	time
Swissmetro	0	0	1	cost	freq.	time

+ alternative specific variance

## Comparison (using 500 draws)

$\mathcal{L}$	Logit		ASV		ASV norm.	
	-5315.39		-5240.414		-5240.414	
	Estim.	Scaled	Estim.	Scaled	Estim.	Scaled
ASC_CAR	0.189	-0.175	0.248	-0.140	0.248	-0.140
ASC_SM	0.451	-0.418	0.900	-0.508	0.901	-0.509
B_COST	-1.08	1.00	-1.77	1.00	-1.77	1.00
B_FR	-5.35	4.95	-7.78	4.40	-7.78	4.40
B_TIME	-1.28	1.19	-1.71	0.966	-1.71	0.966
SIGMA_CAR			0.0107			
SIGMA_TRAIN			0.0284		0.0282	
SIGMA_SM			-3.21		-3.22	

# Identification issue: process

## Examine the variance-covariance matrix

- 1 Specify the model of interest
- 2 Take the **differences** in utilities
- 3 Apply the **order condition**: necessary condition
- 4 Apply the **rank condition**: sufficient condition
- 5 Apply the **equality condition**: verify equivalence

# Heteroscedastic: specification

## Model

$$\begin{aligned}
 U_1 &= \beta x_1 + \sigma_1 \xi_1 && + \nu_1 \\
 U_2 &= \beta x_2 && + \sigma_2 \xi_2 && + \nu_2 \\
 U_3 &= \beta x_3 && + \sigma_3 \xi_3 && + \nu_3 \\
 U_4 &= \beta x_4 && && + \sigma_4 \xi_4 && + \nu_4
 \end{aligned}$$

where  $\xi_i \sim N(0, 1)$ ,  $\nu_i \sim EV(0, \mu)$

## Covariance matrix ( $\gamma = \pi^2/6$ )

$$\text{Cov}(U) = \begin{pmatrix} \sigma_1^2 + \gamma/\mu^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 + \gamma/\mu^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2 + \gamma/\mu^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2 + \gamma/\mu^2 \end{pmatrix}$$

# Heteroscedastic: differences

## Utility differences

$$U_1 - U_4 = \beta(x_1 - x_4) + (\sigma_1\xi_1 - \sigma_4\xi_4) + (\nu_1 - \nu_4)$$

$$U_2 - U_4 = \beta(x_2 - x_4) + (\sigma_2\xi_2 - \sigma_4\xi_4) + (\nu_2 - \nu_4)$$

$$U_3 - U_4 = \beta(x_3 - x_4) + (\sigma_3\xi_3 - \sigma_4\xi_4) + (\nu_3 - \nu_4)$$

## Covariance of utility differences

$\text{Cov}(\Delta U) =$

$$\begin{pmatrix} \sigma_1^2 + \sigma_4^2 + 2\gamma/\mu^2 & \sigma_4^2 + \gamma/\mu^2 & \sigma_4^2 + \gamma/\mu^2 \\ \sigma_4^2 + \gamma/\mu^2 & \sigma_2^2 + \sigma_4^2 + 2\gamma/\mu^2 & \sigma_4^2 + \gamma/\mu^2 \\ \sigma_4^2 + \gamma/\mu^2 & \sigma_4^2 + \gamma/\mu^2 & \sigma_3^2 + \sigma_4^2 + 2\gamma/\mu^2 \end{pmatrix}$$

# Heteroscedastic: order condition

## Upper bound

- $S$  is the number of estimable parameters
- $J$  is the number of alternatives

$$S \leq \frac{J(J-1)}{2} - 1$$

- It represents the number of entries in the lower part of the (symmetric) var-cov matrix
- minus 1 for the scale
- $J = 4$  implies  $S \leq 5$



# Heteroscedastic: rank condition

## Idea

- Number of estimable parameters =
- number of linearly independent equations
- -1 for the scale

$\text{Cov}(\Delta U) =$

$$\begin{pmatrix} \sigma_1^2 + \sigma_4^2 + 2\gamma/\mu^2 & & & \\ \sigma_4^2 + \gamma/\mu^2 & \sigma_2^2 + \sigma_4^2 + 2\gamma/\mu^2 & & \\ \sigma_4^2 + \gamma/\mu^2 & \sigma_4^2 + \gamma/\mu^2 & & \\ & & \sigma_3^2 + \sigma_4^2 + 2\gamma/\mu^2 & \end{pmatrix}$$

dependent
scale

## Heteroscedastic: rank condition

Three parameters out of five can be estimated

Formally...

- 1 Identify unique elements of  $\text{Cov}(\Delta U)$
- 2 Compute the Jacobian wrt  $\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \gamma/\mu^2$
- 3 Compute the rank

$$\begin{pmatrix} \sigma_1^2 + \sigma_4^2 + 2\gamma/\mu^2 \\ \sigma_2^2 + \sigma_4^2 + 2\gamma/\mu^2 \\ \sigma_3^2 + \sigma_4^2 + 2\gamma/\mu^2 \\ \sigma_4^2 + \gamma/\mu^2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$S = \text{Rank} - 1 = 3$$

# Heteroscedastic: equality condition

## Normalization

- We know how many parameters can be identified
- There are infinitely many normalizations
- The normalized model is equivalent to the original one
- Obvious normalizations, like constraining extra-parameters to 0 or another constant, may not be valid

# Heteroscedastic: equality condition

## Error components

$$\begin{aligned}
 U_n &= \beta^T x_n + L_n \xi_n + \nu_n \\
 \text{Cov}(U_n) &= L_n L_n^T + (\gamma/\mu^2) I \\
 \text{Cov}(\Delta_j U_n) &= \Delta_j L_n L_n^T \Delta_j^T + (\gamma/\mu^2) \Delta_j \Delta_j^T
 \end{aligned}$$

## Notations

$$\Delta_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\begin{aligned}
 \text{Cov}(U_n) &= \Omega_n = \Sigma_n + \Gamma_n \\
 &\quad \Omega_n^{\text{norm}} = \Sigma_n^{\text{norm}} + \Gamma_n^{\text{norm}} \\
 \text{Cov}(\Delta_j U_n) &= \Omega_{n\Delta} = \Sigma_{n\Delta} + \Gamma_{n\Delta} \\
 &\quad \Omega_{n\Delta}^{\text{norm}} = \Sigma_{n\Delta}^{\text{norm}} + \Gamma_{n\Delta}^{\text{norm}}
 \end{aligned}$$

# Heteroscedastic: equality condition

The following conditions must hold

- Covariance matrices must be equal

$$\Omega_{n\Delta} = \Omega_{n\Delta}^{\text{norm}}$$

- $\Sigma_n^{\text{norm}}$  must be positive semi-definite

# Heteroscedastic: equality condition

Example with 3 alternatives (index  $n$  dropped)

$$\begin{aligned} U_1 &= \beta x_1 + \sigma_1 \xi_1 && + \nu_1 \\ U_2 &= \beta x_2 && + \sigma_2 \xi_2 && + \nu_2 \\ U_3 &= \beta x_3 && + \sigma_3 \xi_3 && + \nu_3 \end{aligned}$$

$$\text{Cov}(\Delta_3 U) = \Omega_\Delta = \begin{pmatrix} \sigma_1^2 + \sigma_3^2 + 2\gamma/\mu^2 & & \\ & \sigma_2^2 + \sigma_3^2 + 2\gamma/\mu^2 & \\ & & \sigma_2^2 + \sigma_3^2 + 2\gamma/\mu^2 \end{pmatrix}$$

- Parameters:  $\{\sigma_1, \sigma_2, \sigma_3, \mu\}$
- Rank condition:  $S = 2$
- $\mu$  is used for the scale

# Heteroscedastic: equality condition

## Change of variables

- Denote  $\nu_i = \sigma_i^2 \mu^2$  (scaled parameters)
- Normalization condition:  $\nu_3 = K$

$$\Omega_{\Delta} = \begin{pmatrix} (\nu_1 + \nu_3 + 2\gamma)/\mu^2 & \\ (\nu_3 + \gamma)/\mu^2 & (\nu_2 + \nu_3 + 2\gamma)/\mu^2 \end{pmatrix}$$

$$\Omega_{\Delta}^{\text{norm}} = \begin{pmatrix} (\nu_1^N + K + 2\gamma)/\mu_N^2 & \\ (K + \gamma)/\mu_N^2 & (\nu_2^N + K + 2\gamma)/\mu_N^2 \end{pmatrix}$$

where index  $N$  stands for “normalized”

# Heteroscedastic: equality condition

First equality condition:  $\Omega_{\Delta} = \Omega_{\Delta}^{\text{norm}}$

$$\begin{aligned}(\nu_3 + \gamma)/\mu^2 &= (K + \gamma)/\mu_N^2 \\(\nu_1 + \nu_3 + 2\gamma)/\mu^2 &= (\nu_1^N + K + 2\gamma)/\mu_N^2 \\(\nu_2 + \nu_3 + 2\gamma)/\mu^2 &= (\nu_2^N + K + 2\gamma)/\mu_N^2\end{aligned}$$

that is, writing the normalized parameters as functions of others,

$$\begin{aligned}\mu_N^2 &= \mu^2(K + \gamma)/(\nu_3 + \gamma) \\ \nu_1^N &= (K + \gamma)(\nu_1 + \nu_3 + 2\gamma)/(\nu_3 + \gamma) - K - 2\gamma \\ \nu_2^N &= (K + \gamma)(\nu_2 + \nu_3 + 2\gamma)/(\nu_3 + \gamma) - K - 2\gamma\end{aligned}$$



## Heteroscedastic: equality condition

### Second equality condition

$$\Sigma^{\text{norm}} = \frac{1}{\mu_N^2} \begin{pmatrix} \nu_1^N & 0 & 0 \\ 0 & \nu_2^N & 0 \\ 0 & 0 & K \end{pmatrix}$$

must be positive semi-definite, that is

$$\mu_N > 0, \nu_1^N \geq 0, \nu_2^N \geq 0, K \geq 0.$$

Putting everything together, we obtain

$$K \geq \frac{(\nu_3 - \nu_i)\gamma}{\nu_i + \gamma}, \quad i = 1, 2$$

## Heteroscedastic: equality condition

Condition to be verified for the normalization to be valid

$$K \geq \frac{(\nu_3 - \nu_i)\gamma}{\nu_i + \gamma}, \quad i = 1, 2$$

- If  $\nu_3 \leq \nu_i$ ,  $i = 1, 2$ , then the rhs is negative, and any  $K \geq 0$  would do. Typically,  $K = 0$ .
- If not,  $K$  must be chosen large enough
- In practice, always select the alternative with minimum variance.

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# Taste heterogeneity

## Motivation

- Population is heterogeneous
- Taste heterogeneity is captured by segmentation
- Deterministic segmentation is desirable but not always possible
- Distribution of a parameter in the population

## Random parameters

$$U_i = \beta_t T_i + \beta_c C_i + \nu_i$$

$$U_j = \beta_t T_j + \beta_c C_j + \nu_j$$

Let  $\beta_t \sim N(\bar{\beta}_t, \sigma_t^2)$ , or, equivalently,

$$\beta_t = \bar{\beta}_t + \sigma_t \xi, \text{ with } \xi \sim N(0, 1).$$

$$U_i = \bar{\beta}_t T_i + \sigma_t \xi T_i + \beta_c C_i + \nu_i$$

$$U_j = \bar{\beta}_t T_j + \sigma_t \xi T_j + \beta_c C_j + \nu_j$$

If  $\nu_i$  and  $\nu_j$  are i.i.d. EV and  $\xi$  is given, we have

$$P(i|\xi) = \frac{e^{\bar{\beta}_t T_i + \sigma_t \xi T_i + \beta_c C_i}}{e^{\bar{\beta}_t T_i + \sigma_t \xi T_i + \beta_c C_i} + e^{\bar{\beta}_t T_j + \sigma_t \xi T_j + \beta_c C_j}}, \text{ and}$$

$$P(i) = \int_{\xi} P(i|\xi) f(\xi) d\xi.$$

# Random parameters

## Example with Swissmetro

	ASC_CAR	ASC_SBB	ASC_SM	B_COST	B_FR	B_TIME
Car	1	0	0	cost	0	time
Train	0	0	0	cost	freq.	time
Swissmetro	0	0	1	cost	freq.	time

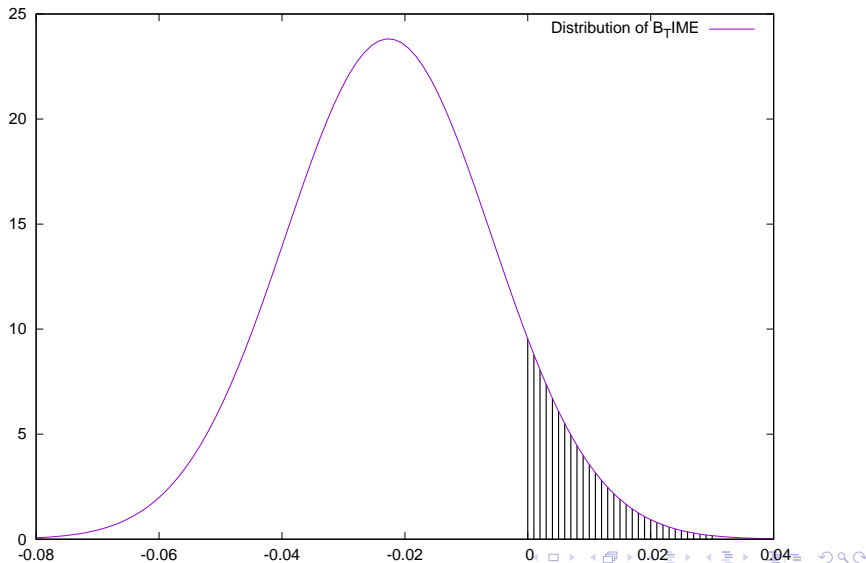
B\_TIME randomly distributed across the population, normal distribution

# Random parameters

## Estimation results

	Logit	RC
$\mathcal{L}$	-5315.4	-5198.0
ASC_CAR_SP	0.189	0.118
ASC_SM_SP	0.451	0.107
B_COST	-0.011	-0.013
B_FR	-0.005	-0.006
B_TIME	-0.013	-0.023
S_TIME		0.017
Prob(B_TIME $\geq$ 0)		8.8%
$\chi^2$		234.84

# Random parameters





# Random parameters

## Example with Swissmetro

	ASC_CAR	ASC_SBB	ASC_SM	B_COST	B_FR	B_TIME
Car	1	0	0	cost	0	time
Train	0	0	0	cost	freq.	time
Swissmetro	0	0	1	cost	freq.	time

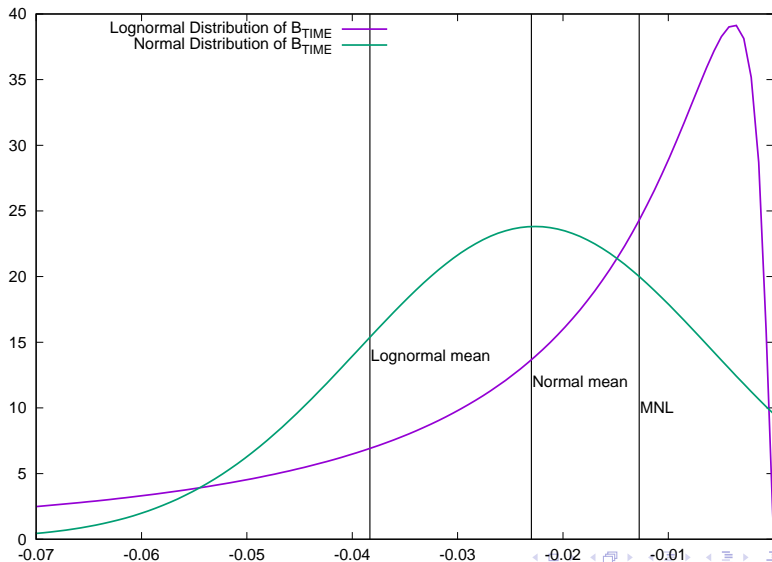
B\_TIME randomly distributed across the population, log normal distribution

# Random parameters

## Estimation results

	Logit	RC-norm.	RC-logn.	
	-5315.4	-5198.0	-5215.81	
ASC_CAR_SP	0.189	0.118	0.122	
ASC_SM_SP	0.451	0.107	0.069	
B_COST	-0.011	-0.013	-0.014	
B_FR	-0.005	-0.006	-0.006	
B_TIME	-0.013	-0.023	-4.033	-0.038
S_TIME		0.017	1.242	0.073
Prob( $\beta > 0$ )		8.8%	0.0%	
$\chi^2$		234.84	199.16	

# Random parameters



# Random parameters

## Example with Swissmetro

	ASC_CAR	ASC_SBB	ASC_SM	B_COST	B_FR	B_TIME
Car	1	0	0	cost	0	time
Train	0	0	0	cost	freq.	time
Swissmetro	0	0	1	cost	freq.	time

B\_TIME randomly distributed across the population, discrete distribution

$$P(\beta_{\text{time}} = \hat{\beta}) = \omega_1 \quad P(\beta_{\text{time}} = 0) = \omega_2 = 1 - \omega_1$$

# Random parameters

## Estimation results

	Logit	RC-norm.	RC-logn.		RC-disc.
	-5315.4	-5198.0	-5215.8		-5191.1
ASC_CAR_SP	0.189	0.118	0.122		0.111
ASC_SM_SP	0.451	0.107	0.069		0.108
B_COST	-0.011	-0.013	-0.014		-0.013
B_FR	-0.005	-0.006	-0.006		-0.006
B_TIME	-0.013	-0.023	-4.033	-0.038	-0.028
					0.000
S_TIME		0.017	1.242	0.073	
W1					0.749
W2					0.251
Prob( $\beta > 0$ )		8.8%	0.0%		0.0%
$\chi^2$		234.84	199.16		248.6

# Outline

- 1 Mixtures
- 2 Relaxing the independence assumption
- 3 Relaxing the identical distribution assumption
- 4 Taste heterogeneity
- 5 Latent classes**
- 6 Summary

# Latent classes

## Capture unobserved heterogeneity

They can represent different:

- Choice sets
- Decision protocols
- Tastes
- Model structures
- etc.

# Latent classes

## Model structure

$$P_n(i|\mathcal{C}_n) = \sum_{s=1}^S P_n(i|\mathcal{C}_n, s)Q_n(s)$$

- $P_n(i|\mathcal{C}_n, s)$  is the class-specific choice model
  - probability of choosing  $i$  given that the individual  $n$  belongs to class  $s$
- $Q_n(s)$  is the class membership model
  - probability of belonging to class  $s$



# Outline

- 1 Mixtures
- 2 Relaxing the independence assumption
- 3 Relaxing the identical distribution assumption
- 4 Taste heterogeneity
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- 6 Summary**

# Summary

## Logit mixtures models

- Computationally more complex than MEV
- Allow for more flexibility than MEV

## Continuous mixtures

Alternative specific variance, nesting structures, random parameters

$$P_n(i) = \int_{\xi} P_n(i|\xi) f(\xi) d\xi$$

## Discrete mixtures

Latent classes of decision makers

$$P_n(i|C_n) = \sum_{s=1}^S P_n(i|C_n, s) Q_n(s)$$

# Tips for applications

- Be careful: simulation can mask specification and identification issues
- Do not forget about the systematic portion

## Appendix: Simulation

How to calculate?

$$P(i) = \int_{\xi} \Pr(i|\xi) f(\xi) d\xi$$

No closed form formula

Monte Carlo simulation

- Randomly draw numbers such that their frequency matches the density  $f(\xi)$
- Let  $\xi^1, \dots, \xi^R$  be these numbers
- The choice model can be approximated by

$$P(i) \approx \frac{1}{R} \sum_{r=1}^R \Pr(i|r), \text{ as } \lim_{R \rightarrow \infty} \frac{1}{R} \sum_{r=1}^R \Pr(i|r) = \int_{\xi} \Pr(i|\xi) f(\xi) d\xi$$

## Appendix: Simulation

### Approximation

$$P(i) \approx \frac{1}{R} \sum_{r=1}^R \Pr(i|r).$$

The kernel is a logit model, easy to compute

$$\Pr(j|r) = \frac{e^{V_{1n+r}}}{e^{V_{1n+r}} + e^{V_{2n+r}} + e^{V_{3n}}}$$

Therefore, it amounts to generating the appropriate draws.

## Appendix: Simulation

### Pseudo-random numbers generators

Although deterministically generated, numbers exhibit the properties of random draws

- Uniform distribution
- Standard normal distribution
- Transformation of standard normal
- Inverse CDF
- Multivariate normal

## Appendix: Simulation

### Uniform distribution

- Almost all programming languages provide generators for a uniform  $U(0, 1)$
- If  $r$  is a draw from a  $U(0, 1)$ , then

$$s = (b - a)r + a$$

is a draw from a  $U(a, b)$

## Appendix: Simulation

### Standard normal

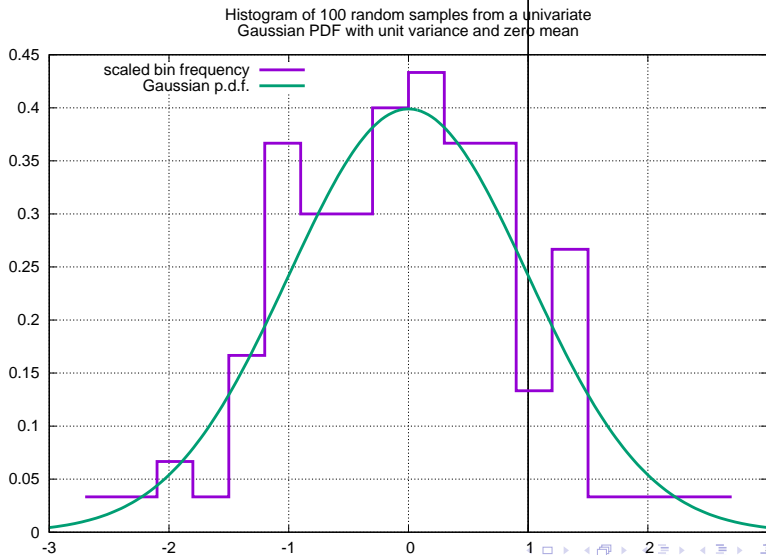
- If  $r_1$  and  $r_2$  are independent draws from  $U(0, 1)$ , then

$$s_1 = \sqrt{-2 \ln r_1} \sin(2\pi r_2)$$
$$s_2 = \sqrt{-2 \ln r_1} \cos(2\pi r_2)$$

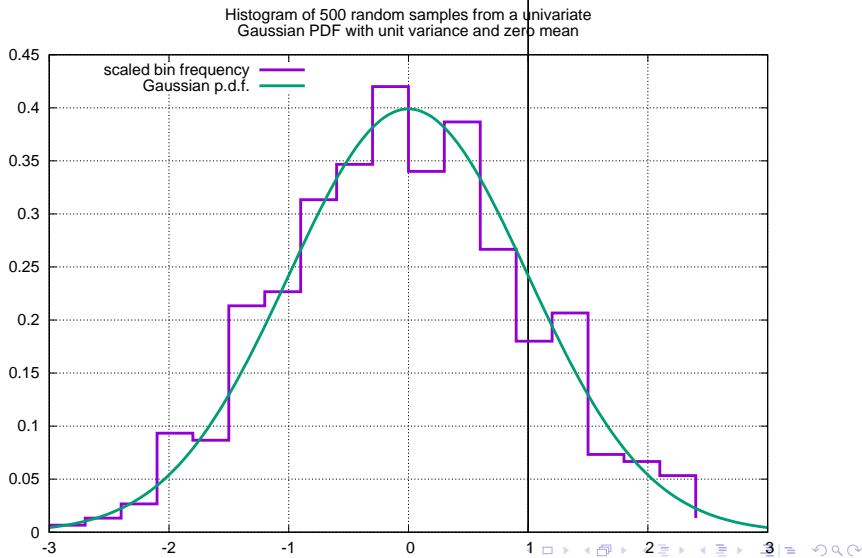
are independent draws from  $N(0, 1)$



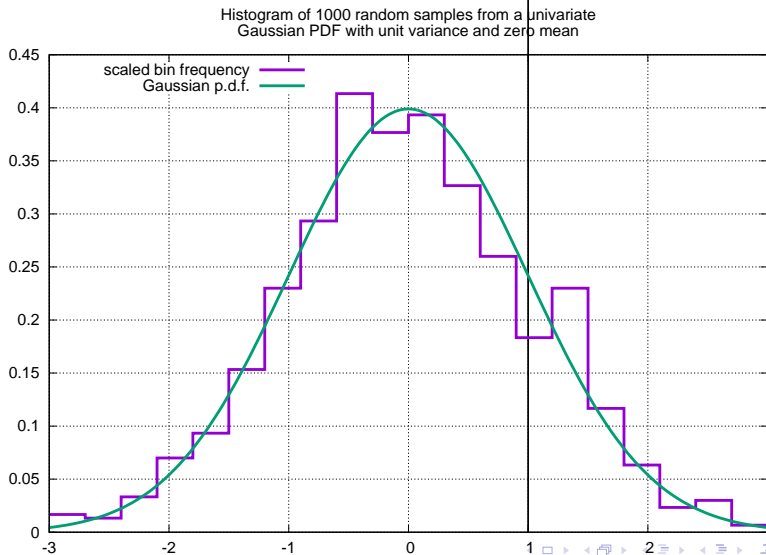
# Appendix: Simulation: standard normal



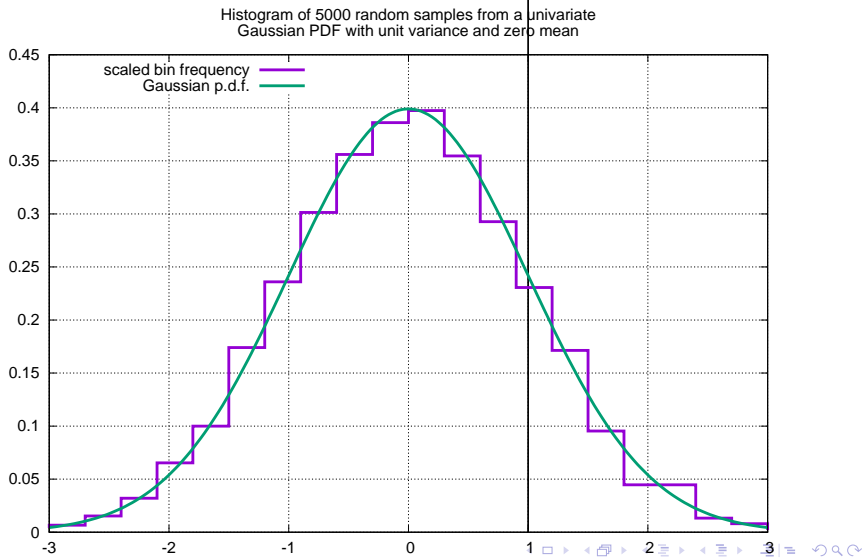
# Appendix: Simulation: standard normal



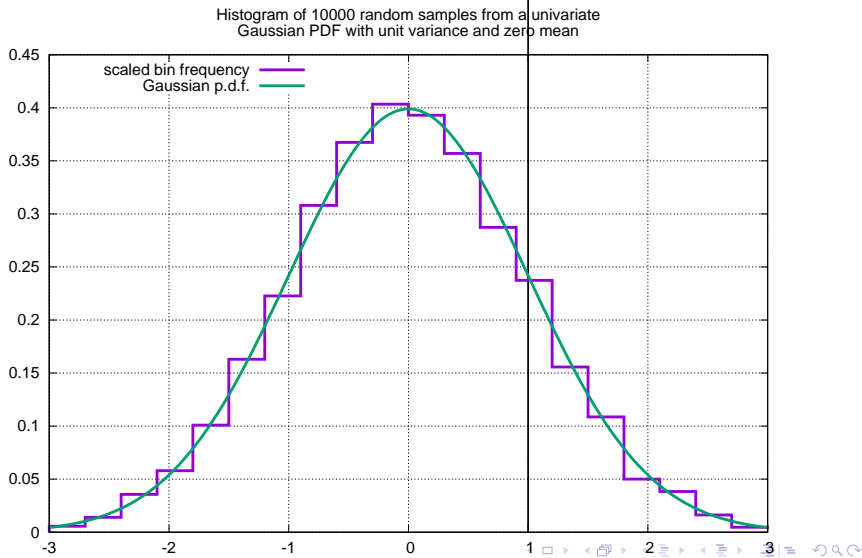
# Appendix: Simulation: standard normal



# Appendix: Simulation: standard normal



# Appendix: Simulation: standard normal



## Appendix: Simulation

### Normal distribution

If  $r$  is a draw from  $N(0, 1)$ , then

$$s = br + a$$

is a draw from  $N(a, b^2)$

### Log normal distribution

If  $r$  is a draw from  $N(a, b^2)$ , then

$$e^r$$

is a draw from a log normal  $LN(a, b^2)$  with mean  $e^{a+(b^2/2)}$  and variance  $e^{2a+b^2}(e^{b^2} - 1)$

## Appendix: Simulation

### Inverse CDF

- Consider a univariate r.v. with CDF  $F(\varepsilon)$
- If  $F$  is invertible and if  $r$  is a draw from  $U(0, 1)$ , then

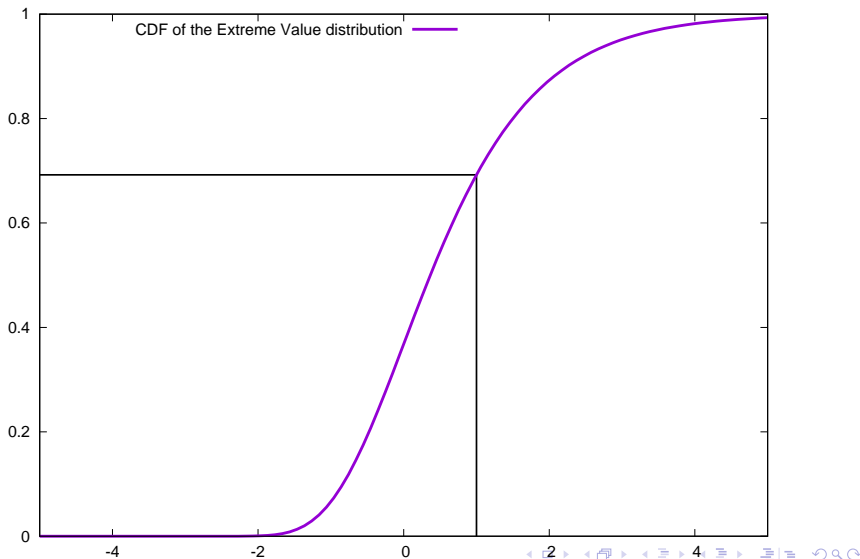
$$s = F^{-1}(r)$$

is a draw from the given r.v.

- Example: EV with

$$F(\varepsilon) = e^{-e^{-\varepsilon}} \quad F^{-1}(r) = -\ln(-\ln r)$$

# Appendix: Simulation: inverse CDF





## Appendix: Simulation

### Multivariate normal

If  $r_1, \dots, r_n$  are independent draws from  $N(0, 1)$ , and

$$r = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$$

then

$$s = a + Lr$$

is a vector of draws from the  $n$ -variate normal  $N(a, LL^T)$ , where

- $L$  is lower triangular, and
- $LL^T$  is the Cholesky factorization of the variance-covariance matrix

# Appendix: Simulation

## Example

$$L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$

$$s_1 = l_{11}r_1$$

$$s_2 = l_{21}r_1 + l_{22}r_2$$

$$s_3 = l_{31}r_1 + l_{32}r_2 + l_{33}r_3$$

# Appendix: Simulation

## Mixtures of logit

$$P(\text{auto}|X) = \int_{\xi} \Pr(\text{auto}|X, \xi) f(\xi) d\xi$$

- Draw from  $f(\xi)$  to obtain  $r_1, \dots, r_R$
- Compute  $P(\text{auto}|X) \approx \tilde{P}(\text{auto}|X) = \frac{1}{R} \sum_{k=1}^R P(\text{auto}|X, r_k) =$

$$\frac{1}{R} \sum_{k=1}^R \frac{e^{\beta X_{\text{auto}} + r_{1k}}}{e^{\beta X_{\text{auto}} + r_{1k}} + e^{\beta X_{\text{bus}} + r_{2k}} + e^{\beta X_{\text{subway}} + r_{3k}}}$$

## Appendix: Maximum simulated likelihood

Solve

$$\max_{\theta} \mathcal{L}(\theta) = \sum_{n=1}^N \left( \sum_{j=1}^J y_{jn} \ln \tilde{P}(j; \theta) \right)$$

where  $y_{jn} = 1$  if ind.  $n$  has chosen alt.  $j$ , 0 otherwise.

Vector of parameters  $\theta$  contains

- usual (fixed) parameters of the choice model
- parameters of the density of the random parameters
- For instance, if  $\beta_j \sim N(\mu_j, \sigma_j^2)$ ,  $\mu_j$  and  $\sigma_j$  are parameters to be estimated

## Appendix: Maximum simulated likelihood

### Warning

- $\tilde{P}(j; \theta)$  is an unbiased estimator of  $P(j; \theta)$

$$E[\tilde{P}_n(j; \theta)] = P(j; \theta)$$

- $\ln \tilde{P}(j; \theta)$  is **not** an unbiased estimator of  $\ln P(j; \theta)$

$$\ln E[\tilde{P}(j; \theta)] \neq E[\ln \tilde{P}(j; \theta)]$$

- Under some conditions, it is a **consistent** (asymptotically unbiased) estimator, so that many draws are necessary.

## Appendix: Maximum simulated likelihood

### Properties of MSL

- If  $R$  is fixed, MSL is inconsistent
- If  $R$  rises at any rate with  $N$ , MSL is consistent
- If  $R$  rises faster than  $\sqrt{N}$ , MSL is asymptotically equivalent to ML.