Decomposition for network design: Exercises 3 **Bernard Gendron**

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1 (50 points)

Consider the following integer programming model, called (LOC):

$$\min \sum_{j\in J} f_j y_j + \sum_{k\in K}\sum_{j\in J} c_{jk} x_{jk},$$

$$\sum_{j\in J} x_{jk} = 1, \quad k \in K, \tag{1}$$

$$x_{jk} \leq y_j, \quad k \in K,\ j \in J,$$

$$x_{jk} \geq 0, \quad k \in K,\ j \in J,$$

$$y_j \in \{0,1\}, \quad j \in J.$$

where $c_{jk} \geq 0$, $k \in K$, $j \in J$, et $f_j \geq 0$, $j \in J$.

1. Formulate the Lagrangian dual corresponding to the Lagrangian relaxation of constraints (1). How would you solve the Lagrangian subproblem?

2. Show that the Lagrangian dual in number 1 can be formulated as follows, using the primal interpretation of Lagrangian duality:

$$\min \sum_{j\in J} \sum_{R\subseteq K|R\neq\emptyset} (f_j + \sum_{k\in R} c_{jk})\lambda_R^j,$$

$$\sum_{j\in J} \sum_{R\subseteq K|k\in R} \lambda_R^j = 1, \quad k \in K,$$

$$\sum_{R\subseteq K|R\neq\emptyset} \lambda_R^j \leq 1, \quad j \in J,$$

$$\lambda_R^j \in [0,1], \quad R \subseteq K|R \neq \emptyset,\ j \in J.$$

3. Show that a reformulation of (LOC) is obtained by adding integrality constraints to the Lagrangian dual model in 2. What is the meaning of variables $\lambda_R^j \in \{0,1\}$, $R \subseteq K|R \neq \emptyset$, $j \in J$?

4. Show that the reformulation of (LOC) derived in 3 is equivalent to the following model:

$$\min \sum_{R\subseteq K|R\neq\emptyset} d_R \lambda_R,$$

$$\sum_{R\subseteq K|k\in R} \lambda_R = 1, \quad k \in K,$$

$$\lambda_R \in \{0,1\}, \quad R \subseteq K|R \neq \emptyset,$$

where $d_R = \min_{j\in J}(f_j + \sum_{k\in R} c_{jk})$.

5. Explain how to solve the LP relaxation of the model in 4 by column generation. In particular, specify an initial master problem and explain how to solve the pricing problem.

2 (50 points)

Consider the uncapacitated network design problem defined over a directed network $G = (N, A)$ with a set of commodities K such that each commodity k has a demand $d^k > 0$ between one origin $O(k)$ and one destination $D(k)$. The problem is to satisfy the demands at minimum cost, given a fixed cost $f_{ij} \geq 0$ for using arc $(i, j) \in A$ and a cost $c_{ij} \geq 0$ per unit of flow on arc $(i, j) \in A$. Consider the following variables: $y_{ij} = 1$, if arc (i, j) is used to carry some flow, 0, otherwise (design variables); x_{ij}^k is the fraction of the demand d^k that flows on arc (i, j). We then obtain the following model:

$$Z = \min \sum_{(i,j)\in A} \sum_{k\in K} d^k c_{ij} x_{ij}^k + \sum_{(i,j)\in A} f_{ij} y_{ij}$$

$$\sum_{j\in N_i^+} x_{ij}^k - \sum_{j\in N_i^-} x_{ji}^k = \begin{cases} 1, & i = O(k) \\ -1, & i = D(k) \\ 0, & i \neq O(k), D(k) \end{cases} \qquad i \in N,\ k \in K \quad (\pi_i^k)$$

$$x_{ij}^k \leq y_{ij} \quad (i, j) \in A,\ k \in K \quad (\beta_{ij}^k)$$

$$x_{ij}^k \geq 0 \quad (i, j) \in A,\ k \in K$$

$$y_{ij} \in \{0, 1\} \quad (i, j) \in A.$$

1. Apply Benders decomposition, the design variables being the “complicating” ones. Write down the Benders subproblem and explain how to solve it.
2. By exploiting the separability of the Benders subproblem, if applicable, write down the dual of the Benders subproblem. Derive the general form of Benders optimality and feasibility cuts.
3. Give a necessary and sufficient condition to ensure the feasibility of the Benders subproblem.
4. Using the condition in number 3, give an equivalent expression of Benders feasibility cuts.
5. Write down the Benders reformulation.