Nested logit models

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Transport and Mobility Laboratory
Red bus/Blue bus paradox

- Mode choice example
- Two alternatives: car and bus
- There are red buses and blue buses
- Car and bus travel times are equal: $T$
Red bus/Blue bus paradox

Model 1

\[ U_{\text{car}} = \beta T + \varepsilon_{\text{car}} \]
\[ U_{\text{bus}} = \beta T + \varepsilon_{\text{bus}} \]

Therefore,

\[ P(\text{car}|\{\text{car, bus}\}) = P(\text{bus}|\{\text{car, bus}\}) = \frac{e^{\beta T}}{e^{\beta T} + e^{\beta T}} = \frac{1}{2} \]
Red bus/Blue bus paradox

Model 2

\[ U_{\text{car}} = \beta T + \varepsilon_{\text{car}} \]
\[ U_{\text{blue bus}} = \beta T + \varepsilon_{\text{blue bus}} \]
\[ U_{\text{red bus}} = \beta T + \varepsilon_{\text{red bus}} \]

\[
P(\text{car}|\{\text{car, blue bus, red bus}\}) = \frac{e^{\beta T}}{e^{\beta T} + e^{\beta T} + e^{\beta T}} = \frac{1}{3}
\]

\[
\begin{align*}
P(\text{car}|\{\text{car, blue bus, red bus}\}) \\
P(\text{blue bus}|\{\text{car, blue bus, red bus}\}) \\
P(\text{red bus}|\{\text{car, blue bus, red bus}\})
\end{align*}
\]

\[
\begin{aligned}
&= \frac{1}{3}.
\end{aligned}
\]
Red bus/Blue bus paradox

- Assumption of MNL: $\varepsilon$ i.i.d
- $\varepsilon_{\text{blue bus}}$ and $\varepsilon_{\text{red bus}}$ contain common unobserved attributes:
  - fare
  - headway
  - comfort
  - convenience
  - etc.
Capturing the correlation

- Bus
- Car
- Blue
- Red
Capturing the correlation

If bus is chosen then

\[
\begin{align*}
U_{\text{blue bus}} &= V_{\text{blue bus}} + \varepsilon_{\text{blue bus}} \\
U_{\text{red bus}} &= V_{\text{red bus}} + \varepsilon_{\text{red bus}}
\end{align*}
\]

where \( V_{\text{blue bus}} = V_{\text{red bus}} = \beta T \)

\[
P(\text{blue bus}|\{\text{blue bus, red bus}\}) = \frac{e^{\beta T}}{e^{\beta T} + e^{\beta T}} = \frac{1}{2}
\]
Capturing the correlation
Capturing the correlation

What about the choice between bus and car?

\[ U_{\text{car}} = \beta T + \varepsilon_{\text{car}} \]
\[ U_{\text{bus}} = V_{\text{bus}} + \varepsilon_{\text{bus}} \]

with

\[ V_{\text{bus}} = V_{\text{bus}}(V_{\text{blue bus}}, V_{\text{red bus}}) \]
\[ \varepsilon_{\text{bus}} = ? \]

Define \( V_{\text{bus}} \) as the expected maximum utility of red bus and blue bus.
Expected maximum utility

For a set of alternative $C$, define

$$U_C = \max_{i \in C} U_i = \max_{i \in C} (V_i + \varepsilon_i)$$

and

$$V_C = E[U_C]$$

For MNL

$$E[\max_{i \in C} U_i] = \frac{1}{\mu} \ln \sum_{i \in C} e^{\mu V_i} + \frac{\gamma}{\mu}$$
Expected maximum utility

\[
V_{bus} = \frac{1}{\mu_b} \ln(e^{\mu_b V_{blue\ bus}} + e^{\mu_b V_{red\ bus}}) \\
= \frac{1}{\mu_b} \ln(e^{\mu_b \beta T} + e^{\mu_b \beta T}) \\
= \beta T + \frac{1}{\mu_b} \ln 2
\]

where \(\mu_b\) is the scale parameter for the MNL associated with the choice between red bus and blue bus.
Nested Logit Model

Probability model:

\[
P(\text{car}) = \frac{e^{\mu V_{\text{car}}}}{e^{\mu V_{\text{car}}} + e^{\mu V_{\text{bus}}}} = \frac{e^{\mu \beta T}}{e^{\mu \beta T} + e^{\mu \beta T} + \frac{\mu}{\mu_b} \ln 2} = \frac{1}{1 + 2 \frac{\mu}{\mu_b}}
\]

If \( \mu = \mu_b \), then \( P(\text{car}) = \frac{1}{3} \) (Model 2)
If \( \mu_b \to \infty \), then \( \frac{\mu}{\mu_b} \to 0 \), and \( P(\text{car}) \to \frac{1}{2} \) (Model 1)

Note for \( \mu_b \to \infty \)

\[
e^{\mu V_{\text{bus}}} = \frac{1}{2} e^{\mu V_{\text{red bus}}} + \frac{1}{2} e^{\mu V_{\text{blue bus}}}
\]
Nested Logit Model

Probability model:

\[ P(\text{bus}) = \frac{e^{\mu V_{\text{bus}}}}{e^{\mu V_{\text{car}}} + e^{\mu V_{\text{bus}}}} = \frac{e^{\mu \beta T + \frac{\mu}{\mu_b} \ln 2}}{e^{\mu \beta T} + e^{\mu \beta T + \frac{\mu}{\mu_b} \ln 2}} = \frac{1}{1 + 2^{-\frac{\mu}{\mu_b}}} \]

If \( \mu = \mu_b \), then \( P(\text{bus}) = \frac{2}{3} \) (Model 2)

If \( \frac{\mu}{\mu_b} \to 0 \), then \( P(\text{bus}) \to \frac{1}{2} \) (Model 1)
Nested Logit Model
Solving the paradox

If $\frac{\mu}{\mu_b} \to 0$, we have

\[
\begin{align*}
P(\text{car}) &= 1/2 \\
P(\text{bus}) &= 1/2 \\
P(\text{red bus}|\text{bus}) &= 1/2 \\
P(\text{blue bus}|\text{bus}) &= 1/2 \\
P(\text{red bus}) &= P(\text{red bus}|\text{bus})P(\text{bus}) = 1/4 \\
P(\text{blue bus}) &= P(\text{blue bus}|\text{bus})P(\text{bus}) = 1/4
\end{align*}
\]
- A group of similar alternatives is called a nest
- Each alternative belongs to exactly one nest
- The model is named Nested Logit
- The ratio $\mu/\mu_b$ must be estimated from the data
- $0 < \mu/\mu_b \leq 1$ (between models 1 and 2)
A case study

- Choice of a residential telephone service
- Household survey conducted in Pennsylvania, USA, 1984
- Revealed preferences
- 434 observations
A case study

Availability of telephone service by residential area:

<table>
<thead>
<tr>
<th>Budget Measured</th>
<th>Adjacent to Metro area</th>
<th>Adjacent to metro area</th>
<th>Other non-metro areas</th>
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</thead>
<tbody>
<tr>
<td>Standard Measured</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Local Flat</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Extended Area Flat</td>
<td>no</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Metro Area Flat</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>
Multinomial Logit Model

\[ C = \{\text{BM}, \text{SM}, \text{LF}, \text{EF}, \text{MF}\} \]

\[
\begin{align*}
V_{\text{BM}} &= \beta_{\text{BM}} + \beta_c \ln(\text{cost}_{\text{BM}}) \\
V_{\text{SM}} &= \beta_{\text{SM}} + \beta_c \ln(\text{cost}_{\text{SM}}) \\
V_{\text{LF}} &= \beta_{\text{LF}} + \beta_c \ln(\text{cost}_{\text{LF}}) \\
V_{\text{EF}} &= \beta_{\text{EF}} + \beta_c \ln(\text{cost}_{\text{EF}}) \\
V_{\text{MF}} &= \beta_c \ln(\text{cost}_{\text{MF}})
\end{align*}
\]

\[
P(i|C) = \frac{e^{V_i}}{\sum_{j \in C} e^{V_j}}
\]
## Multinomial Logit Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MNL</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>Value</td>
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<tr>
<td>$\beta_{BM}$</td>
<td>-2.46</td>
</tr>
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</tr>
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<tr>
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</tr>
<tr>
<td>$\beta_c$</td>
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</tr>
<tr>
<td>$\mathcal{L}_0$</td>
<td>-560.25</td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
<td>-477.56</td>
</tr>
<tr>
<td># Obs</td>
<td>434</td>
</tr>
</tbody>
</table>
Nested Logit Model

- Measured
  - BM
  - SM

- Flat
  - LF
  - EF
  - MF

Nested logit models – p.21/38
Nested Logit Model

Model of the choice among “measured” alternatives

\[ P(i|M) = \frac{e^{V_i}}{e^{V_{BM}} + e^{V_{SM}}} \quad i = BM, SM \]

We estimate the model with the 196 observations choosing either BM or SM, and calculate the inclusive value

\[ I_M = \ln(e^{V_{BM}} + e^{V_{SM}}) \]

for all observations (scale normalized to 1)
### Nested Logit Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MNL</th>
<th>Measured</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value</td>
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</tr>
<tr>
<td>$\beta_{BM}$</td>
<td>-2.46</td>
<td>(-7.84)</td>
</tr>
<tr>
<td>$\beta_{SM}$</td>
<td>-1.74</td>
<td>(-6.28)</td>
</tr>
<tr>
<td>$\beta_{LF}$</td>
<td>-0.54</td>
<td>(-2.57)</td>
</tr>
<tr>
<td>$\beta_{EF}$</td>
<td>-0.74</td>
<td>(-1.02)</td>
</tr>
<tr>
<td>$\beta_c$</td>
<td>-2.03</td>
<td>(-9.47)</td>
</tr>
<tr>
<td>$L_0$</td>
<td>-560.3</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>-477.6</td>
<td></td>
</tr>
<tr>
<td># Obs</td>
<td>434</td>
<td></td>
</tr>
</tbody>
</table>
Nested Logit Model

Model of the choice among “flat” alternatives

\[ P(i|M) = \frac{e^{V_i}}{e^{V_{LF}} + e^{V_{EF}} + e^{V_{MF}}} \quad i = \text{LF, EF, MF} \]

We estimate the model with the 238 observations choosing LF, EF or MF and calculate the inclusive value

\[ I_F = \ln(e^{V_{LF}} + e^{V_{EF}} + e^{V_{MF}}) \]

for all observations (scale normalized to 1)
## Nested Logit Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MNL</th>
<th>Measured</th>
<th>Flat</th>
</tr>
</thead>
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<tr>
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<td>Value</td>
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<tr>
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<td>$\beta_{SM}$</td>
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<td>$\beta_{LF}$</td>
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<tr>
<td># Obs</td>
<td>434</td>
<td>196</td>
<td>238</td>
</tr>
</tbody>
</table>
Nested Logit Model

Model of the choice of type of service

\[ P(M) = \frac{e^{\mu(\tilde{\beta}_M + I_M)}}{e^{\mu(\tilde{\beta}_M + I_M)} + e^{\mu I_F}} = \frac{e^{\beta_M + \mu I_M}}{e^{\beta_M + \mu I_M} + e^{\mu I_F}} \]

\[ P(F) = \frac{e^{\mu I_F}}{e^{\mu(\tilde{\beta}_M + I_M)} + e^{\mu I_F}} = \frac{e^{\mu I_F}}{e^{\beta_M + \mu I_M} + e^{\mu I_F}} \]

- \( I_M \) and \( I_F \) are attributes of measured and flat, resp.
- \( \beta_M = \mu \tilde{\beta}_M \) and \( \mu \) are unknown parameters, to be estimated.
- \( 0 < \mu \leq 1 \)
## Nested Logit Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MNL</th>
<th>Measured</th>
<th>Flat</th>
<th>Nested Logit</th>
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<td></td>
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<tr>
<td>(\beta_{BM})</td>
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<tr>
<td>(\beta_{SM})</td>
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<tr>
<td>(\beta_{LF})</td>
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<tr>
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<tr>
<td>(\mu)</td>
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<td>434</td>
<td></td>
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</tr>
</tbody>
</table>
Nested Logit Model

How to interpret the log-likelihood?
Assume that individual $n$ has chosen alt. $i$ in nest $M$.

$$P_n(i) = P_n(i|M)P_n(M)$$

Consider now all individuals choosing an alt. $i$ in nest $M$

$$\sum_n \ln P_n(i) = \sum_n \ln P_n(i|M) + \sum_n \ln P_n(M) = \mathcal{L}_M + \sum_n \ln P_n(M)$$

For individuals choosing an alternative $j$ in nest $F$, we have

$$\sum_n \ln P_n(j) = \sum_n \ln P_n(i|F) + \sum_n \ln P_n(F) = \mathcal{L}_F + \sum_n \ln P_n(F)$$
Nested Logit Model

Therefore, we obtain that

\[ \mathcal{L} = \mathcal{L}_M + \mathcal{L}_F + \mathcal{L}_{NL} \]
## Nested Logit Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MNL Value</th>
<th>MNL (t-stat)</th>
<th>Measured Value</th>
<th>Measured (t-stat)</th>
<th>Flat Value</th>
<th>Flat (t-stat)</th>
<th>Nested Logit Value</th>
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<tr>
<td>$\beta_{BM}$</td>
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Nested Logit Model

Which value of $\beta_c$ should we use?

Measured: -3.12 (-4.76) or Flat: -3.73 (-6.22)

Equal $\beta_c$’s:

- Jointly estimate measured and flat models and constrain $\beta_C$ to be equal
- Declare “Measured” alternatives unavailable when a “Flat” alternative is chosen, and vice versa.
## Nested Logit Model

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<tr>
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<th>Nested Logit</th>
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Nested Logit Model

Multinomial Logit:

\[ P(BM) = \frac{e^{V_{BM}}}{\sum_{j \in C} e^{V_j}} \]

Nested Logit:

\[
P(BM) = P(BM|M)P(M) = \frac{e^{V_{BM}}}{e^{V_{BM} + e^{V_{SM}}}} \cdot \frac{e^{\beta_M + \mu I_M}}{e^{\beta_M + \mu I_M + e^{V_{IF}}}} \cdot \frac{e^{\beta_M + \mu \ln(e^{V_{BM} + e^{V_{SM}}})}}{e^{\beta_M + \mu \ln(e^{V_{LM} + e^{V_{SM}}}) + e^{V_{MF}}}}
\]
Nested Logit Model

Let \( \mu = 1 \)

\[
P(BM) = \frac{e^{V_{BM}}}{e^{V_{BM}} + e^{V_{SM}}} e^{\beta_M \ln(e^{V_{BM}} + e^{V_{SM}})} + e^{\ln(e^{V_{LF}} + e^{V_{EF}} + e^{V_{MF}})}
\]

\[
= \frac{e^{V_{BM}}}{e^{V_{BM}} + e^{V_{SM}}} e^{\beta_M \left(e^{V_{BM}} + e^{V_{SM}}\right)} + e^{V_{LF}} + e^{V_{EF}} + e^{V_{MF}}
\]

\[
= \frac{e^{V_{BM}}}{e^{V_{BM}} + e^{V_{SM}} + e^{V_{LF} - \beta_M} + e^{V_{EF} - \beta_M} + e^{V_{MF} - \beta_M}}
\]
Nested Logit Model

In general, if $C = \bigcup_{m=1,\ldots,M} C_m$,

$$P(i|C_m) = \frac{e^{\mu_m V_i}}{\sum_{j \in C_m} e^{\mu_m V_i}} \quad \text{and} \quad P(C_m|C) = \frac{e^{\mu V'_m}}{\sum_{k=1,\ldots,m} e^{\mu V'_k}}$$

where

$$V'_m = \frac{1}{\mu_m} \ln \sum_{i \in C_m} (e^{\mu_m V_i})$$

When $\frac{\mu}{\mu_m} = 1$, for all $m$, NL becomes MNL
Simultaneous estimation

\[ P(i|C) = P(i|C_m)P(C_m|C) \]

Note that each \( i \) belongs to exactly one nest \( m \) i.e. the \( C_m \)'s do not overlap

The log-likelihood for observation \( n \) is

\[ \ln P(i_n|C_n) = \ln P(i_n|C_{mn}) + \ln P(C_{mn}|C_n) \]

where \( i_n \) is the chosen alternative.
Simultaneous estimation

Sequential estimation:
- Estimation of NL decomposed into two estimations of MNL
- Estimator is consistent but not efficient

Simultaneous estimation:
- Log-likelihood function is generally non concave
- No guarantee of global maximum
- Estimator asymptotically efficient
Simultaneous estimation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MNL</th>
<th>Seq. Nested Logit</th>
<th>Sim. Nested Logit</th>
</tr>
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<tbody>
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</tr>
<tr>
<td>$\beta_{SM}$</td>
<td>-1.74</td>
<td>(-6.28)</td>
<td>0.79</td>
</tr>
<tr>
<td>$\beta_{LF}$</td>
<td>-0.54</td>
<td>(-2.57)</td>
<td>-1.07</td>
</tr>
<tr>
<td>$\beta_{EF}$</td>
<td>-0.74</td>
<td>(-1.02)</td>
<td>-1.28</td>
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<tr>
<td>$\beta_{c}$</td>
<td>-2.03</td>
<td>(-9.47)</td>
<td>-3.47</td>
</tr>
<tr>
<td>$\beta_{M}$</td>
<td> </td>
<td> </td>
<td>-1.66</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.42</td>
<td>(5.85)</td>
<td>0.46</td>
</tr>
<tr>
<td>$L_0$</td>
<td>-560.3</td>
<td> </td>
<td>-566.2</td>
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<tr>
<td>$L$</td>
<td>-477.6</td>
<td> </td>
<td>-473.6</td>
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<tr>
<td># Obs</td>
<td>434</td>
<td> </td>
<td>434</td>
</tr>
</tbody>
</table>

Compare $\beta_{M} = -1.66$ and $\mu \beta_{BM} = -1.74$

Compare $\beta_{SM} - \beta_{BM} = 0.79$ for Seq. and Sim.